

Variational Principle and Lagrangian Formulation

In mechanical problems that are complex, owing to the presence of various forces of constraint, it becomes quite difficult to obtain a solution using Newton's equations. In order to effect a simplification, alternative procedural methods are required. One such method, employing calculus of variations, is variational principle which will form the subject matter of this chapter.

2.0. INTRODUCTION TO CALCULUS OF VARIATIONS

Solution of a dynamical problem means that we want to locate the position of the system, *e.g.*, a particle, at a particular instant of time. We are also interested in the *path* adopted by the system. The *piece-wise* information of this path *i.e.* where it is maximum or minimum is secured through differential calculus by putting $\dot{y}(x) = 0$ etc. But if we want the information about the *whole* path then we shall have to look for integral calculus and will be interested in the arguments like whether the path as a whole is largest or shortest (extremum or having a stationary value). This study requires the technique of calculus of variations. Refer to fig. (2.0). Two paths which a particle may follow in going from position 1 to position 2 are shown. The straight line path is shortest and can be represented as

$$y = mx + c$$

$$y = y(x)$$

or in functional form indicating that y is a function of independent parameter x . For each value of x there will be a *fixed* value of y . As we are interested in the length of the path, denoted by say, I , we can write

$$\begin{aligned} I &= \int dS = \int \sqrt{dx^2 + dy^2} \\ &= \int \sqrt{1 + \dot{y}^2} dx \\ &= \int f(\dot{y}) dx \\ &= \int f(y, \dot{y}, x) dx, \end{aligned}$$

because $\dot{y}$ involves both y and x , we have expressed it in f . Thus we note that path, I , is the integral of the function, f , which itself is the function of y . If we want that path, I , be extremum, any variation, δ , in it, *i.e.*, δI should be zero. That is

$$\delta I = \delta \int f(y, \dot{y}, x) dx = 0$$

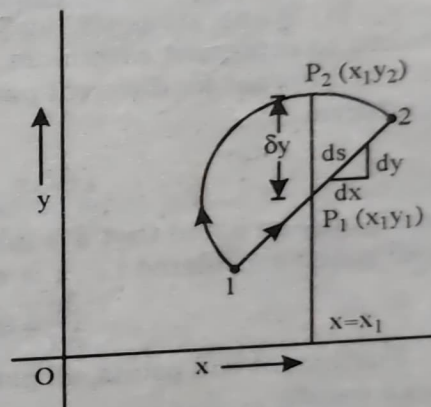


Fig. 2.0.

$$= \delta \int f[y(x), \dot{y}(x), x] dx = 0.$$

This is the formulation of the problem of calculus of variations. Note that variation δ is defined as the variation in the quantity to which it is applied at the fixed value of independent parameter *i.e.*

$$\delta y = (y_2 - y_1)_{x=x_1}$$

If we account for both curves then the equation $y = y(x)$ is incapable to represent both, *e.g.* $y = mx + c$ will represent straight line path only. Therefore to include other paths, we require another parameter, say α , to designate paths, to be introduced in y *i.e.*

$$y = y(x, \alpha)$$

so that

$$y = y(x, \alpha_1)$$

and

$$y = y(x, \alpha_2)$$

may represent the two paths. However, the relationship can be expressed as

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x).$$

where $\eta(x)$ is any arbitrary function of x which vanishes at end points. Note $\alpha = 0$ in $y(x, 0)$ may be taken to represent extremum path. Thus α represents the paths. It means I , the path length, which is different for different paths, will also be a function of α , *i.e.*, $I = I(\alpha)$ so that we may write the integral

$$I(\alpha) = \int f[y(x, \alpha), \dot{y}(x, \alpha), x] dx,$$

Thus it may be noted that δ is taken at a fixed value of independent parameter, *i.e.*, x is the same for all paths considered *i.e.*, x is not the function of α or we can write

$$\frac{\partial x}{\partial \alpha} = 0.$$

Further, at end points, all paths meet and therefore there is no variation even in y co-ordinate at end points

$$\text{or} \quad \left. \frac{\partial y}{\partial \alpha} \right|_{\text{end points 1 \& 2}} = 0.$$

The shortest path is taken as the extremum path and the other paths with which we compare it (in our discussion of finding the extremum path) are called comparison paths. We shall discuss in later articles that path of a particle in configuration space of Lagrangian formulation is uncertain and consequently a discussion about extremum path is an important consideration for the system in motion.

More about δ -Variation Process :

We would like here to mention a bit more about δ -variation process. As has been said earlier, it represents the increase in the quantity to which it is applied as we pass from extremum path to the comparison path at the same value of independent variable. Thus δy represents arbitrary variation of $y(x)$ with respect to an arbitrary parameter, α about its extremum value. In other words, δy corresponds to a *virtual displacement*. Such a virtual displacement does not always coincide with possible actual displacement occurring in the course of motion, *i.e.*, varied path may or may not be an actual path. The two important points in this variation, significant to note, are;

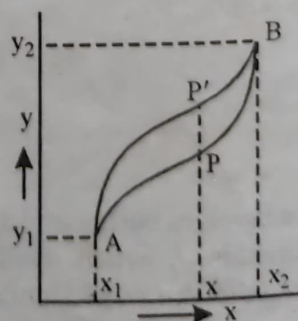


Fig. 2.1. δ -variation.

(i) end points' co-ordinates are same for each curve which gives $\left. \frac{\partial y}{\partial \alpha} \right|_{\text{end points}} = 0$.

Further, variation is considered at a fixed value of independent co-ordinate x ; $\frac{\partial x}{\partial \alpha} = 0$ for such a variation.

(ii) the system must travel from one end point A to other end point B in the same time for all the paths conceived. It also predicts that end points' time is also fixed for every path. This gives that $\frac{\partial t}{\partial \alpha} = 0$ along the path and also at end points. To satisfy these conditions, we regard that system point (Art 2.3) moves with different velocities along different paths so as to keep end points' time fixed, i.e., there is no variation of time along any path, i.e., $\delta t = 0$.

2.1. SOME TECHNIQUES OF THE CALCULUS OF VARIATIONS

Calculus of variations deals with like problems: we find a function $y(x)$, such that a definite integral taken over the function of this function, say $f(y)$, will assume a stationary value. Thus we wish to find a function $y(x)$ which will cause the integral

$$I = \int_{x_1}^{x_2} f(y, \dot{y}, x) dx \quad \dots (1)$$

to have a stationary value (extremum). The integrand f is taken to be a function of the dependent variable y , independent variable x and $\dot{y} = \left(\frac{dy}{dx} \right)$. Thus we write

$$\delta I = \delta \int_{x_1}^{x_2} f(y, \dot{y}, x) dx = 0,$$

where δ is called variation and represents the increase in the quantity to which it is applied in switching from stationary path to the comparison path at the fixed value of x .

Since $y(x)$ represents a path, our aim is to find a curve (path) between two points (x_1, y_1) and (x_2, y_2) for which the integral is an extremum. We will, in solving the problem, take into account all the possible paths between the two points x_1 and x_2 . We label all the possible curves $y(x)$ with different value of a parameter α regarding that for some value of α , say $\alpha = 0$, the curves would coincide with the path or paths for which the integral $\int f(y, \dot{y}, x) dx$ is extremum

(may be taken as the shortest). Therefore, y should then be the function of both, x and the parameter α , i.e.

$$y(x, \alpha) = y(x, 0) + \alpha \eta(x),$$

where $\eta(x)$ is any arbitrary function of x which vanishes at $x = x_1$ and $x = x_2$ since it is variation with fixed ends.

Thus integral I , which will now be a function of α , can be written as

$$I(\alpha) = \int_{x_1}^{x_2} f[y(x, \alpha), \dot{y}(x, \alpha), x] dx.$$

Since α refers to different paths, $\left(\frac{\partial I}{\partial \alpha} \right) = 0$ will correspond to a path for which the integral is an extremum. Thus

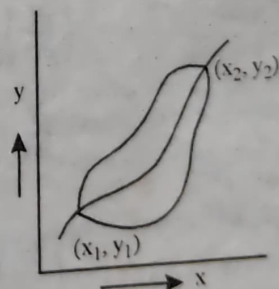


Fig. 2.2. Possible paths of motion.

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} + \frac{\partial f}{\partial x} \frac{\partial x}{\partial \alpha} \right\} dx.$$

Since x is not a function of α , $\frac{\partial x}{\partial \alpha} = 0$, so that

$$\begin{aligned} \frac{\partial I(\alpha)}{\partial \alpha} &= \int_{x_1}^{x_2} \left\{ \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} \right\} dx \\ &= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial \alpha} dx \\ &= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}} \frac{\partial^2 y}{\partial x \partial \alpha} dx. \end{aligned}$$

Integrating w.r.to x the second term by parts,

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} \frac{\partial y}{\partial \alpha} dx + \left. \frac{\partial f}{\partial \dot{y}} \frac{\partial y}{\partial \alpha} \right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} dx. \quad \dots (2)$$

Since the end points (x_1, y_1) and (x_2, y_2) are *fixed* and *same* for every curve so at end points x_1 and x_2 , differentiation of different paths of motion vanishes, i.e. $\frac{\partial y}{\partial \alpha} = 0$. Thus eq. (2) reduces to

$$\frac{\partial I}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right] \frac{\partial y}{\partial \alpha} dx.$$

To find the stationary value, we multiply the above equation by $d\alpha$ and evaluate the derivative at $\alpha = 0$, so that

$$\left(\frac{\partial I}{\partial \alpha} \right)_0 d\alpha = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right] \left(\frac{\partial y}{\partial \alpha} \right)_0 d\alpha dx \quad \dots (3)$$

But

$$\left(\frac{\partial I}{\partial \alpha} \right)_0 d\alpha = \text{increase in the integral } I \text{ as we pass from the extremum path to the comparison path at the same value of } x \text{ (i.e., } \delta\text{-variation of } I) = \delta I.$$

Similarly

$$\left(\frac{\partial y}{\partial \alpha} \right)_0 d\alpha = \delta y.$$

Therefore equation (3) takes the form

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right] \delta y dx.$$

But $\delta I = 0$, then

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) \right] \delta y dx = 0.$$

Here δy represents some arbitrary variation of $y(x)$ with respect to the arbitrary parameter α about its extremum value ($\alpha = 0$). Since δy is arbitrary,

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0, \quad \dots (4)$$

which is a relation that should be satisfied by a function f [which is a function of $y(x)$] if the integral I is to be extremum. Readers would note that equations derived in the next article called Lagrange's equations of motion, bear a great similarity with eq. (4).

FEW EXAMPLES :

(A) **Shortest distance between two points in a plane : Geodesics** : A straight line is regarded as the shortest distance between two points in a plane. Thus straight line is an extremum path of a particle in a plane and consequently, the equation of such a path should conveniently be obtained from the above technique of the calculus of variations.

An element of small arc length ds in a plane can be represented as

$$ds = \sqrt{(dx^2 + dy^2)} = dx \sqrt{(1 + \dot{y}^2)}$$

The total length of a curve between any two points 1 and 2 in a plane can be written as

$$I = \int_1^2 ds = \int_1^2 \sqrt{(1 + \dot{y}^2)} dx = \int_1^2 f dx.$$

For this curve to be shortest, $\delta I = 0$, i.e., the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0, \quad \dots (1)$$

where $f = \sqrt{(1 + \dot{y}^2)}$ must be satisfied.

Now if

$$f = \sqrt{(1 + \dot{y}^2)}, \quad \frac{\partial f}{\partial y} = 0,$$

and

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{(1 + \dot{y}^2)}}.$$

The equation (1) therefore reduces to

$$\frac{d}{dx} \left[\frac{\dot{y}}{\sqrt{(1 + \dot{y}^2)}} \right] = 0,$$

or

$$\frac{\dot{y}}{\sqrt{(1 + \dot{y}^2)}} = \text{a constant}$$

from which

$$\dot{y} = a,$$

where a is constant, giving

$$y = ax + b.$$

This is an equation representing a straight line; a and b can be calculated in terms of end points' coordinates of the curve.

(B) **Minimum surface of revolution** : We form a surface of revolution by revolving a curve about a certain axis. In this example, curve passing through two end points (x_1, y_1) and (x_2, y_2) has been rotated about y -axis. Our aim is to find a curve which on revolving about a certain axis forms geometry of minimum surface area.

Let us consider a strip at point A formed due to the revolution of arc length ds about y axis. If distance of this arc from y -axis be x , then the surface area of strip

$$= 2\pi x ds$$

$$= 2\pi x \sqrt{(1 + \dot{y}^2)} dx.$$

The total surface area is then

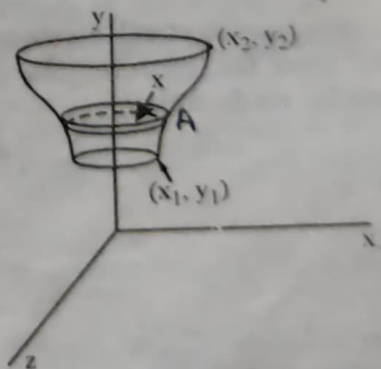


Fig. (2.3a). Minimum surface of revolution.

$$I = \int_1^2 2\pi x \sqrt{1 + \dot{y}^2} dx$$

and will be minimum if $\delta I = 0$, for which the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0$$

should be satisfied. For this problem $f = x \sqrt{1 + \dot{y}^2}$, so that

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

Putting in the said equation, we get

$$\frac{d}{dx} \left(\frac{x\dot{y}}{\sqrt{1 + \dot{y}^2}} \right) = 0$$

or

$$\frac{x\dot{y}}{\sqrt{1 + \dot{y}^2}} = a, \text{ constant of integration.}$$

Squaring and arranging, we get

$$\dot{y}^2 = \frac{a^2}{x^2 - a^2}$$

the integration of which yields

$$y = a \int \frac{dx}{\sqrt{x^2 - a^2}} + b = a \cosh^{-1} \frac{x}{a} + b$$

or

$$x = a \cosh \frac{y - b}{a}$$

which is the equation of a catenary.

(C) The brachistochrone problem : In this problem, we find a curve joining two points along which a particle falling from rest under the influence of gravity travels from higher to the lower point in the minimum time.

Suppose v is the speed of the particle along the curve, then in traversing ds portion of the curve, time spent will be ds/v so that total time taken by the particle in moving from highest point 1 to the lowest point 2 will be

$$t_{12} = \int_1^2 \frac{ds}{v}$$

Suppose vertical distance of fall upto the point 2 be x , then from the conservation principle for the energy of the particle, we find that

$$\frac{1}{2} mv^2 = mgx$$

or

$$v = \sqrt{2gx}$$

Then

$$t_{12} = \int_1^2 \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gx}} = \int_1^2 \frac{\sqrt{1 + \dot{y}^2}}{\sqrt{2gx}} dx$$

$$t_{12} = \int_1^2 f dx$$

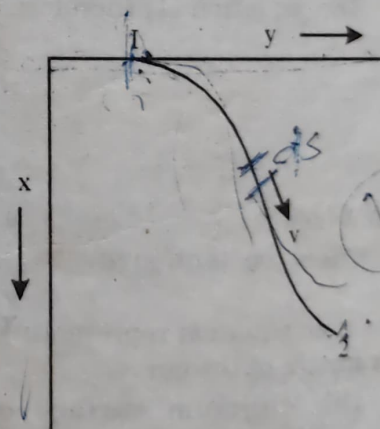


Fig. (2.3b)

where

$$f = \left(\frac{1 + \dot{y}^2}{2gx} \right)^{1/2}$$

For t_{12} to be minimum, the equation

$$\frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) - \frac{\partial f}{\partial y} = 0$$

must be satisfied. From the expression for f , find that

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{(2gx)} \sqrt{(1 + \dot{y}^2)}}$$

so that above equation becomes

$$\frac{d}{dx} \left(\frac{\dot{y}}{\sqrt{(2gx)} \sqrt{(1 + \dot{y}^2)}} \right) = 0,$$

or

$$\frac{\dot{y}^2}{x(1 + \dot{y}^2)} = C,$$

where C is a constant. On rearranging the above equation, we find

$$\frac{\dot{y}^2}{C} = x(1 + \dot{y}^2)$$

or

$$\dot{y}^2 \left(\frac{1}{C} - x \right) = x$$

or

$$\dot{y}^2 \left(\frac{x}{C} - x^2 \right) = x^2$$

or

$$\dot{y} = \frac{x}{\left(\frac{x}{C} - x^2 \right)^{1/2}}$$

Let us put

$$\frac{1}{C} = 2a,$$

so that on integration,

$$y = a \cos^{-1} \left(1 - \frac{x}{a} \right) - (2ax - x^2)^{1/2} + C',$$

where C' is the new constant of integration.

Obviously, if C' be zero then y will be zero for x to be zero. In such a case,

$$y = a \cos^{-1} \left(1 - \frac{x}{a} \right) - (2ax - x^2)^{1/2}$$

which represents an inverted cycloid with its base along y -axis and cusp at the origin.

2.2. VARIATIONAL TECHNIQUE FOR MANY INDEPENDENT VARIABLES, EULER-LAGRANGE DIFFERENTIAL EQUATION

Technique of the previous article is now to be generalised. Let f be a function of many independent variables y_j and their derivatives $\dot{y}_j$. Then the integral I , representing a path between two points 1 and 2 will be written as

$$I = \int_1^2 f[y_1(x), y_2(x), \dots, \dot{y}_1(x), \dot{y}_2(x), \dots, x] dx. \quad \dots (1)$$

Now to account for all possible curves between the two points 1, 2; we, merely for procedure, assign different values of a parameter α to these curves, so that y_j will also be a function of α , i.e., curves being represented by $y_j(x, \alpha)$. The family of curves may be represented as

* The cycloid is the curve traced by a pt. on the circumference of a circle which rolls (without sliding) on a straight line.

$$y_1(x, \alpha) = y_1(x, 0) + \alpha \eta_1(x),$$

$$y_2(x, \alpha) = y_2(x, 0) + \alpha \eta_2(x).$$

where η_1, η_2 etc. are completely arbitrary functions of x which vanish at end points and the curves $y_1(x, 0), y_2(x, 0)$ etc. for $\alpha = 0$ are paths for which the integral I is extremum.

The integral I will now be a function of α and hence its variation can be represented as

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_1^2 \sum_j \left(\frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \dot{y}_j} \frac{\partial \dot{y}_j}{\partial \alpha} d\alpha \right) dx.$$

Integrating by parts the second term of the integrand we get

$$= \int_1^2 \sum_j \left(\frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial \alpha} d\alpha \right) dx + \sum_j \frac{\partial f}{\partial \dot{y}_j} \frac{\partial y_j}{\partial \alpha} dx \Big|_1^2 - \int_1^2 \sum_j \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_j} \right) \frac{\partial y_j}{\partial \alpha} d\alpha dx.$$

Since at end points, which are held fixed, all paths meet, $\frac{\partial y_j}{\partial \alpha} \Big|_1^2 = 0$. Therefore,

$$\begin{aligned} \frac{\partial I}{\partial \alpha} d\alpha &= \int_1^2 \sum_j \left(\frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial \alpha} d\alpha \right) dx - \int_1^2 \sum_j \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_j} \right) \frac{\partial y_j}{\partial \alpha} d\alpha dx \\ &= \int_1^2 \sum_j \left[\left(\frac{\partial f}{\partial y_j} \right) - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_j} \right) \right] \frac{\partial y_j}{\partial \alpha} d\alpha dx. \end{aligned}$$

Like in art. 2.1, we put

$$\frac{\partial I}{\partial \alpha} d\alpha = \delta I,$$

$$\frac{\partial y_j}{\partial \alpha} d\alpha = \delta y_j,$$

so that

$$\delta I = \int_1^2 \sum_j \left[\left(\frac{\partial f}{\partial y_j} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_j} \right) \right) \right] \delta y_j dx.$$

For the integral I to be extremum,

$$\int_1^2 \sum_j \left[\left(\frac{\partial f}{\partial y_j} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_j} \right) \right) \right] \delta y_j dx = 0.$$

Since δy_j are independent of each other, coefficients of δy_j should separately vanish if above equation is to be satisfied. Thus

$$\left[\frac{\partial f}{\partial y_j} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}_j} \right) \right] = 0, \quad j = 1, 2, \dots, n. \quad \dots (2)$$

The set of differential equations represented by (2) are known as Euler-Lagrange differential equations. Thus solutions of Euler-Lagrange differential equation represent those curves for

which the integral $I = \int_1^2 f(y_j, \dot{y}_j, x) dx$ assumes an extremum value.

2.3. HAMILTON'S VARIATIONAL PRINCIPLE

System Point and Configuration Space : We shall first explain the meaning of the 'motion of the system' as it is involved in Hamilton's variational principle. The determination of the motion

of a single particle in three dimensions is a mechanical problem. (The mechanical problem involving two particles, every particle being described by a set of three co-ordinates, can be reduced to a single particle problem simply by regarding that the single particle moves in a six dimensional space. Thus, in general, a problem involving N -particles can be treated as one of a 'single particle' moving along a trajectory in $3N$ -dimensional space. This space is referred to as 'configuration space' and the single particle as 'system point'. Motion of 'system point' in 'configuration space' is called the motion of the system between any two given instants. Configuration space has no necessary connection with the real three dimensional space.

The principle states that the integral $\int_{t_1}^{t_2} (T - V) dt$ shall have a stationary value (extremum),

where T , the kinetic energy of the mechanical system, is a function of co-ordinates and their derivatives and V , the potential energy of the mechanical system, is a function of co-ordinates only. Such a system for which V is purely a function of co-ordinates is called conservative system.

Statement : *Hamilton's principle.*

Hamilton's principle for conservative system is stated as follows :

The motion of the system from time t_1 to time t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L dt,$$

where $L = T - V$, is an extremum for the path of motion.

Deduction :

Let us consider a conservative system of particles. Employing the generalised co-ordinates the integral can be written as

$$\int_{t_1}^{t_2} [T(q_j, \dot{q}_j) - V(q_j)] dt.$$

Therefore, according to Hamilton's principle, we have

$$\delta \int_{t_1}^{t_2} [T(q_j, \dot{q}_j) - V(q_j)] dt = 0$$

$$\text{or} \quad \int_{t_1}^{t_2} \sum_j \left[\left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right) - \frac{\partial V}{\partial q_j} \delta q_j \right] dt = 0$$

$$\text{or} \quad \int_{t_1}^{t_2} \sum_j \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \right) \delta q_j dt + \int_{t_1}^{t_2} \sum_j \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j dt = 0$$

$$\text{or} \quad \int_{t_1}^{t_2} \sum_j \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \right) \delta q_j dt + \int_{t_1}^{t_2} \sum_j \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} (\delta q_j) dt = 0$$

Integrating, by parts the second term, we get

$$\int_{t_1}^{t_2} \sum_j \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \right) \delta q_j dt + \sum_j \frac{\partial T}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j dt = 0$$

Since in such a variation, there is no co-ordinate variation at end points, $\delta q_j \Big|_{t_1}^{t_2} = 0$. Hence equation reduces to

$$\int_{t_1}^{t_2} \sum_j \left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial q_j} \right) \delta q_j dt - \int_{t_1}^{t_2} \sum_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j dt = 0$$

$$\int_{t_1}^{t_2} \sum_j \left[\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \right] \delta q_j dt = 0.$$

Since each δq_j are independent of each other, the coefficient of every δq_j should be equated to zero to satisfy above equation. Thus equating the coefficient of j^{th} co-ordinate to zero, we get

$$\left[\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \right] = 0.$$

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j} (T - V) \right] = 0.$$

Now, for conservative systems, V is not a function of velocities $\dot{q}_j$ but only of co-ordinates. Therefore

$$\left[\frac{d}{dt} \left(\frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j} (T - V) \right] = 0.$$

Here we introduce the concept of a scalar function, called *Lagrangian* L for a conservative system and is equal to $(T - V)$. L is a scalar function of q_j , $\dot{q}_j$ and t . The q_j variables constitute the basis vectors of an abstract $3N$ -dimensional configuration space. Thus the above equation takes the form

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0.$$

This set of equations is called Lagrange's equations of motion. Each is a *second order differential equation* in terms of the time as independent variable.

As is obvious, motion of the system has been described by the energy consideration and not by that of force. Further, since Lagrangian, L , is purely a scalar, the solution of these equations of motion is easy as compared to Newton's equations of motion which involve the concept of force and hence bear a vectorial sense. As each Lagrange's equation is second order equation, its solution involves only two arbitrary constants. *The form of the equation is independent of the co-ordinates used.*

We shall see in next articles that Lagrangian L is not unique in its functional form because it is possible to preserve the form of Lagrange's equations of motion for a variety of choices of the Lagrangian. If a Lagrangian is given, we can construct other equivalent Lagrangians. For any classical holonomic system, L can be constructed either from $T - V$ or from $T - U$ (V refers to potential energy for conservative and U for non-conservative systems) as the case may be. But in special relativistic dynamics the classical definition of $L = T - V$ or $L = T - U$ is no longer valid.

2.4. DEDUCTION OF LAGRANGE'S EQUATIONS OF MOTION FROM HAMILTON'S PRINCIPLE (For conservative systems)

Though this deduction is already given in previous article but here we want to proceed directly with Lagrangian $L(q_j, \dot{q}_j, t)$.

According to Hamilton's variational principle, motion of a conservative system from time t_1 to time t_2 is such that the variation of the line integral

$$I = \int_{t_1}^{t_2} L[q_j(t), \dot{q}_j(t), t] dt$$

is zero, i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t) dt = 0. \quad \dots (1)$$

Now we shall show that the Lagrange's equations of motion follow directly from Hamilton's principle. If we account for all possible paths of motion of the system in configuration space and label each with a value of a parameter α , then, since paths are being represented by $q_j(t, \alpha)$, I also becomes a function of α so that we write

$$I(\alpha) = \int_{t_1}^{t_2} L[q_j(t, \alpha), \dot{q}_j(t, \alpha), t] dt$$

so that

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \alpha} + \frac{\partial L}{\partial t} \frac{\partial t}{\partial \alpha} \right] dt \quad \dots (2)$$

Since in δ variation, there is no time variation along any path and also at end points and hence $\partial t / \partial \alpha$ is zero along all paths. Therefore on multiplying by $d\alpha$, above equation is

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \alpha} d\alpha dt + \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial^2 q_j}{\partial t \partial \alpha} d\alpha dt \quad \dots (3)$$

Integrating second term by parts,

$$= \int_{t_1}^{t_2} \sum_j \frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial \alpha} d\alpha dt + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \alpha} d\alpha \bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_j \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \frac{\partial q_j}{\partial \alpha} d\alpha dt.$$

The middle term is zero since δ variation involves fixed end points i.e., $\frac{\partial q_j}{\partial \alpha} \bigg|_{t_1}^{t_2}$ is zero at end points t_2 and t_1 . So

$$\frac{\partial I(\alpha)}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \frac{\partial q_j}{\partial \alpha} d\alpha dt$$

or

$$\delta I(\alpha) = \int_{t_1}^{t_2} \sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] \delta q_j dt. \quad \dots (4)$$

Since q_j are independent of each other, the variations δq_j will be independent. Hence $\delta I(\alpha) = 0$ if and only if the coefficients of δq_j separately vanish, i.e.,

$$\sum_j \left[\frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \right] = 0.$$

or

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0. \quad \dots (5)$$

which are Lagrange's equations of motions for a conservative system. It is thus obvious that these equations follow directly from Hamilton's principle.

About the path : Specification of Lagrangian function is equivalent to fixing the equations of motion. If we want any information from them about the path adopted by the system during its motion, it is necessary to provide *six initial values* for a particle—*three for co-ordinates and three for velocities*. But in configuration space, specification of the position of a 'single point' provides only three initial values for each particle and hence *specification of the path is not possible*. Thus, if the

equations of motion are defined, there are an infinite number of possible paths through any point in configuration space*. Due to this reason, path of the motion in configuration space will not have any necessary resemblance to the path in space of any actual particle. Each point on the path represents the *entire* system configuration at some given instant of time. In fact, *configuration space is a verbal device introduced for displaying the motion of a system when employing the Lagrangian approach.*

2.5. DEDUCTION OF LAGRANGE'S EQUATIONS BY DIFFERENTIAL METHOD — D'ALEMBERT'S PRINCIPLE

In Art. 2.4 we have discussed integral method to obtain Lagrange's equations. For the sake of convenience and also for comparative study, we now discuss this differential method, although the previous method is of prime importance as that provides Lagrangian formulation an independent status in mechanics.

(a) D'Alembert's principle :

This method is based on the *principle of virtual work*. The system is subjected to an infinitesimal displacement consistent with the forces and constraints imposed on the system at the given instant t . This *change in the configuration of the system is not associated with a change in time, i.e.,* there is no actual displacement during which forces and constraints may change and hence the displacement is termed as virtual displacement.

Now suppose the system is in equilibrium, *i.e.,* total force $\mathbf{F}_i$ on every particle is zero; then work done by this force in a small virtual displacement $\delta \mathbf{r}_i$ will also vanish. *i.e.,* for whole system for N particles

$$\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0.$$

Let this total force be expressed as sum of applied force $\mathbf{F}_i^a$ and forces of constraints $\mathbf{f}_i$. Then above equation takes the form

$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

We now consider the system for which the virtual work of the forces of constraints is zero. An example of such a system can be had in mind if we consider that a particle be constrained to move on a smooth surface so that the forces of constraints being perpendicular to the surface while virtual displacement tangential to it, then virtual work done by forces of constraints will be zero. Thus

$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{r}_i = 0. \quad \dots (1)$$

The equation is termed as principle of virtual work) To interpret the equilibrium of the system,

* Let us illustrate the point :

If we solve second order equation $\frac{d^2 y}{dx^2} = 0$, equation of a straight line is obtained, *viz.,* $y = mx + c$. Obviously it has two constants of integration. If dy/dx is defined, m is defined (like velocity $\dot{q}_j$) and if y is defined c will also be defined (like q_j). Now if m and c both are fixed, straight line will be fixed or specified. For fixing two constants m and c , we require two initial conditions. Suppose only one initial condition is provided, c (intercept on y -axis). Obviously they will be parallel lines. If, instead, value of c is specified but not of m , there is again the possibility of a number of lines with same c but different m (slope). Equation (5), on solving, leads to the equation of path of system and to fix that path three initial values of $\dot{q}_j$ and three initial values of co-ordinates q_j are required; but since configuration space provides only the three initial values of co-ordinates, path will not be fixed but arbitrary.

D'Alembert adopted an idea of a reversed force. He conceived that a system will remain in equilibrium under the action of a force equal to the actual force $\mathbf{F}_i$ plus reversed effective force $\dot{\mathbf{p}}_i$. Thus

$$\mathbf{F}_i + (-\dot{\mathbf{p}}_i) = 0$$

or

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0.$$

where $-\dot{\mathbf{p}}_i$ appears as an effective force called the *reversed force of inertia* on the i th particle, supplementing the already existing externally applied force $\mathbf{F}_i$.

Thus the principle of virtual work takes the form

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

Again writing

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i,$$

$$\sum_i (\mathbf{F}_i^a - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

Since force of constraints are no more in picture, it is better to drop superscript a . Thus

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \quad \dots (2)$$

which is called *D'Alembert's Principle*. The key point is that in eq. (2) we got rid of the forces of constraints. To satisfy eq. (2), we can not equate the coefficients of $\delta \mathbf{r}_i$ to zero since $\delta \mathbf{r}_i$ are not independent of each other and hence it is necessary to transform $\delta \mathbf{r}_i$ changes into the changes of generalised co-ordinates, δq_j , which are independent of each other. The coefficient of every δq_j will then be equated to zero.

Thus we note that, as this principle does not involve forces of constraints in any way, it is sufficient to specify all the applied forces only. Further it is valid for all rheonomic and scleronomous systems that are either holonomic or homogeneous non-holonomic. The inertial force $-\dot{\mathbf{p}}_i$ is introduced to reduce the problem of dynamics to one of statics. Thus force can be regarded as an inertial force arising in an accelerated frame of reference e.g. in a freely falling accelerated frame $\mathbf{F}_i - \dot{\mathbf{p}}_i = 0$, gravity is nullified by the inertial force even though the whole system is in motion and not in either static or dynamic equilibrium.

(b) Lagrange's Equations from D'Alembert's Principle :

The co-ordinate transformation equations are

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n, t),$$

so that

$$\frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial q_1} \frac{dq_1}{dt} + \frac{\partial \mathbf{r}_i}{\partial q_2} \frac{dq_2}{dt} + \dots + \frac{\partial \mathbf{r}_i}{\partial t} \frac{dt}{dt}$$

or

$$\mathbf{v}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \quad \dots (3)$$

Further infinitesimal displacement $\delta \mathbf{r}_i$ can be connected with δq_j as

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j + \frac{\partial \mathbf{r}_i}{\partial t} \delta t$$

But last term is zero since in virtual displacement only co-ordinate displacement is considered and not that of time. Therefore

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

Now we write eq. (2) as
$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0,$$

$$\sum_{ij} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j - \sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0.$$

We define the term

$$\sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = Q_j,$$

the components of the generalised force. As discussed under 'generalised force' art. 1.10, q 's need not have the dimensions of length, and similarly, it is not necessary for the Q 's to have the dimensions of force, but it is necessary that the product $Q_j \delta q_j$ must have the dimensions of work. Thus above equation takes the form

$$\sum_j Q_j \delta q_j - \sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = 0. \quad \dots (4)$$

Let us evaluate the second term of above eq. (4) :

$$\begin{aligned} \sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j &= \sum_{ij} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\ &= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right\} \delta q_j \\ &= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \mathbf{v}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right\} \delta q_j \quad \dots (5) \end{aligned}$$

Further

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) &= \sum_k \frac{\partial}{\partial q_k} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \frac{dq_k}{dt} + \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \frac{dt}{dt} \\ &= \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_j} \\ &= \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t} \\ &= \frac{\partial}{\partial q_j} \left(\frac{d\mathbf{r}_i}{dt} \right) = \frac{\partial \mathbf{v}_i}{\partial q_j} \quad \dots (6) \end{aligned}$$

Also differentiating eq. (3) with respect to q_j , we get

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \quad \dots (7)$$

Putting eqs. (6) and (7) in eq. (5), we get

$$\begin{aligned} \sum_{ij} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j &= \sum_{ij} \left\{ \frac{d}{dt} \left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right\} \delta q_j \\ &= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j. \end{aligned}$$

With this substitution, eq. (4) becomes

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = 0, \dots$$

where for $\sum_i \frac{1}{2} m_i v_i^2$, T is written since it represents the total kinetic energy of the system. Further

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0.$$

Since the constraints are holonomic, q_j are independent of each other and hence to satisfy above equation the coefficient of each δq_j should separately vanish, i.e.,

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] = Q_j. \quad \dots (8)$$

As j ranges from 1 to N , there will be N such second order equations.

Case I. Conservative System :

Now we fit this problem to conservative system so that above set of equations may be identified as Lagrange's equations of motion. For a conservative system, forces $\mathbf{F}_i$ are derivable from potential function V , the latter being purely dependent on co-ordinates, i.e.,

$$\begin{aligned} \mathbf{F}_i &= -\nabla_i V \\ &= -\frac{\partial V}{\partial \mathbf{r}_i} \end{aligned}$$

Then generalised force at once can be expressed as

$$\begin{aligned} Q_j &= \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\ &= -\sum_i \frac{\partial V}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\ &= -\frac{\partial V}{\partial q_j}. \end{aligned}$$

Eq. (8) is now

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] = -\frac{\partial V}{\partial q_j}$$

or

$$\left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} \right] = 0$$

or

$$\left[\frac{d}{dt} \left(\frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} \right] = 0,$$

since V is not a function of $\dot{q}_j$. Recognising $(T - V)$ as L , the Lagrangian for the conservative system, we see that equations become

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0, \quad \dots (9)$$

which are known as Lagrange's equations of motion (of the second kind) for conservative system.

Case II. Non-conservative System :

If potentials are velocity dependent, called generalised potentials, then though the system is not conservative, yet the above form of Lagrange's equations can be retained provided Q_j , the components of generalised force, are obtained (see eq. 3 of art. 2.6) from a function $U(q_j, \dot{q}_j)$ such that

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right). \quad \dots (10)$$

Putting this value of Q_j in equation (8), we get

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right)$$

or

$$\frac{d}{dt} \left(\frac{\partial (T - U)}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j} (T - U) = 0.$$

If we take Lagrangian $L = T - U$, where U is generalised potential, then above equation is

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0,$$

which is exactly of the same form as eq. (9). An example of such a type will follow in the calculation of Lagrangian for the case of electromagnetic forces on moving charges (see art 2.6-2).

Case III. System involving non-potential forces :

Case I & II both include potential force component derivable either from a velocity independent ordinary potential V or a velocity dependent generalised potential U . There are forces which are not derivable from a potential function; for example Coriolis force $\mathbf{F} = 2m (\mathbf{v} \times \vec{\omega})$ and the Lorentz force on a charged particle moving in a magnetic field $\mathbf{F}_m = q (\mathbf{v} \times \mathbf{B})$. They are the examples of *gyroscopic forces*. The power associated with these forces is zero i.e. $\mathbf{F} \cdot \mathbf{v} = 0$.

If we denote nonpotential forces by Q'_j then Lagrange's equations of motion can be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q'_j \quad (j = 1 \dots n)$$

where

$$Q'_j = Q'_j(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t)$$

in general and $L = T - V$ or $L = T - U$ as the case may be.

We employ Newton's force laws to deduce Lagrange's equations of motion by this differential method. From this deduction, it at once comes to the mind that such a formulation is a direct consequence of Newton's law. Integral method (variational principle), however, maintains that Hamilton's principle, which follows directly from Lagrange's equations, (we have shown earlier the converse, since Hamilton's principle is both a necessary and a sufficient condition for Lagrange's equations of motion) as a parallel fundamental postulate of mechanics. Thus variational principle provides Lagrangian formulation a fundamental base, although variation δq_j involved in variational principle are quite identical with virtual displacements of the co-ordinates encountered in differential method, since no variation of time is involved.

2.6-1. DISSIPATIVE SYSTEM : RAYLEIGH'S DISSIPATION FUNCTION

It can be shown that if a system involves frictional forces or in general dissipative forces, then in suitable circumstance, such a system can also be described in terms of extended Lagrangian formulation.

Frictional forces are found to be proportional to the velocity of the particle so that in cartesian co-ordinates components are

$$F_j^d = -k_j \dot{x}_j,$$

where k_j are constants. Such frictional forces are defined in terms of a new quantity called Rayleigh dissipation function given as

$$\mathfrak{D} = \frac{1}{2} \sum_j k_j \dot{x}_j^2,$$

which yields

$$F_j^d = -\frac{\partial \mathfrak{D}}{\partial \dot{x}_j}.$$

Writing equation (8) in cartesian co-ordinates, assuming that this still holds for such a system,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j.$$

where L contains the potential of conservative forces as described in previous cases; Q_j represents the forces which do not arise from potential *i.e.*,

$$Q_j^d = F_j^d \\ = -\frac{\partial \mathfrak{L}}{\partial \dot{x}_j}$$

Thus equation can be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} + \frac{\partial \mathfrak{L}}{\partial \dot{x}_j} = 0.$$

Expressing Q_j in terms of generalised co-ordinates, we find

$$Q_j^d = \sum_i F_i^d \frac{\partial x_i}{\partial q_j} = - \sum_i k_i \dot{x}_i \frac{\partial x_i}{\partial q_j} \\ = - \sum_i k_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \quad \left(\text{because } \frac{\partial x_i}{\partial q_j} = \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \right) \\ = \frac{\partial}{\partial \dot{q}_j} \left\{ -\frac{1}{2} \sum_i k_i \dot{x}_i^2 \right\} \\ = -\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}.$$

Lagrange's equations now become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} + \frac{\partial \mathfrak{L}}{\partial \dot{q}_j} = 0.$$

Thus for such a system, to obtain equations of motion, two scalars L and $\mathfrak{L}$ are to be specified.

2.6-2. LAGRANGIAN FOR A CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD : GYROSCOPIC FORCES

In an electromagnetic field the force on a particle of charge, q , is given by

$$\mathbf{F} = q \left[\mathbf{E} + \frac{1}{c} (\mathbf{v} \times \mathbf{B}) \right],$$

in which magnetic force $f = q/c (\mathbf{v} \times \mathbf{B})$ is gyroscopic in nature.

But to incorporate it in Lagrangian formulation, we must express $\mathbf{F}$ in terms of vector and scalar potentials $\mathbf{A}$ and ϕ . Thus writing

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.$$

Equation for $\mathbf{F}$ becomes

$$\mathbf{F} = q \left[-\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c} (\mathbf{v} \times \nabla \times \mathbf{A}) \right]. \quad \dots (1)$$

To write it in a more convenient form, let us write the x -component of all terms on right hand side, *i.e.*,

$$(\nabla\phi)_x = \frac{\partial \phi}{\partial x}$$

and

$$(\mathbf{v} \times \nabla \times \mathbf{A})_x = v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - v_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \\ = v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z},$$

on adding and subtracting the term $v_x \frac{\partial A_x}{\partial x}$,

$$(\mathbf{v} \times \nabla \times \mathbf{A})_x = \frac{\partial}{\partial x} (\mathbf{v} \cdot \mathbf{A}) - \left[v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \right] \quad \dots (2)$$

Further, total time derivative of A_x is

$$\begin{aligned} \frac{dA_x}{dt} &= \frac{\partial A_x}{\partial t} + \left(v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \right) \\ \frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} &= v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} \end{aligned}$$

Putting above in eq. (2), we get

$$(\mathbf{v} \times \nabla \times \mathbf{A})_x = \frac{\partial}{\partial x} (\mathbf{v} \cdot \mathbf{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$$

Eq. (1) is now

$$\begin{aligned} F_x &= q \left[-\frac{\partial \phi}{\partial x} - \frac{1}{c} \frac{\partial A_x}{\partial t} + \frac{1}{c} \left\{ \frac{\partial}{\partial x} (\mathbf{v} \cdot \mathbf{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \right\} \right] \\ &= q \left[-\frac{\partial}{\partial x} \left(\phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right) - \frac{1}{c} \frac{dA_x}{dt} \right] \end{aligned}$$

Also

$$\frac{\partial}{\partial v_x} (\mathbf{v} \cdot \mathbf{A}) = A_x, \quad \text{since } \frac{\partial A_x}{\partial v_x} = 0.$$

Thus

$$F_x = q \left[-\frac{\partial}{\partial x} \left(\phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right) + \frac{d}{dt} \left\{ \frac{\partial}{\partial v_x} \left(-\frac{\mathbf{v} \cdot \mathbf{A}}{c} \right) \right\} \right]$$

Since ϕ is independent of velocity, we can write the term

$$\frac{\partial}{\partial v_x} \left(-\frac{\mathbf{v} \cdot \mathbf{A}}{c} \right) \text{ as } \frac{\partial}{\partial v_x} \left(\phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right)$$

Therefore,

$$F_x = q \left[-\frac{\partial}{\partial x} \left(\phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right) + \frac{d}{dt} \left\{ \frac{d}{dx} \left(\phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right) \right\} \right]$$

Let us put

$$U = q\phi - \frac{q}{c} \mathbf{v} \cdot \mathbf{A}.$$

we get

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{x}} \right) \quad \dots (3)$$

It shows that *force acting on the particle is derivable from a potential dependent on velocity*. Such a particle system is called non-conservative one and we apply the following equations of motion

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j. \quad \dots (4)$$

Writing equation (3) in generalised co-ordinates

$$F_x = Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right)$$

and putting in eqs. (4), we get

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right)$$

or

$$\frac{d}{dt} \left[\frac{\partial (T - U)}{\partial \dot{q}_j} \right] - \frac{\partial (T - U)}{\partial q_j} = 0,$$

which can be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0,$$

where Lagrangian

$$L = T - U = T - q\phi + \frac{q}{c} \mathbf{v} \cdot \mathbf{A} \quad T = \frac{1}{2} m \mathbf{v}^2 \quad V = q\phi - \frac{q}{c} \mathbf{v} \cdot \mathbf{A} \quad \dots (5)$$

This shows that the form of Lagrange's equation remains same even if the system is non-conservative. Eq. (5) will be used in further articles involving discussion on non-conservative systems.

2.7-1. DEDUCTION OF HAMILTON'S PRINCIPLE FROM D'ALEMBERT'S PRINCIPLE

D'Alembert's principle is

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i \quad \dots (1)$$

or

Left hand term is

$$\begin{aligned} \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i &= m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \\ &= \frac{d}{dt} \left(m_i \frac{d\mathbf{r}_i}{dt} \right) \cdot \delta \mathbf{r}_i \\ &= \frac{d}{dt} \left(m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \mathbf{r}_i \right) - m_i \frac{d\mathbf{r}_i}{dt} \cdot \frac{d}{dt} (\delta \mathbf{r}_i) \end{aligned} \quad \dots (2)$$

We know that δ can be interpreted as an operator which produces a small change in $\mathbf{r}_i$ not connected with time. It can be treated like a differential operator; both δ , d operators can commute, i.e.,

$$\delta(dx) = d(\delta x).$$

So

$$\delta \left(\frac{d\mathbf{r}_i}{dt} \right) = \frac{d}{dt} (\delta \mathbf{r}_i).$$

Eq. (2) is therefore

$$\begin{aligned} \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i &= \frac{d}{dt} \left(m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \mathbf{r}_i \right) - m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \left(\frac{d\mathbf{r}_i}{dt} \right) \\ &= \frac{d}{dt} \left(m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \mathbf{r}_i \right) - \delta \left[\frac{1}{2} m_i \left(\frac{d\mathbf{r}_i}{dt} \right)^2 \right] \\ &= \frac{d}{dt} \left(m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \mathbf{r}_i \right) - \delta \left(\frac{1}{2} m_i v_i^2 \right). \end{aligned}$$

Putting this in eq. (1), we get

$$\frac{d}{dt} \left[\sum_i \left(m_i \frac{d\mathbf{r}_i}{dt} \right) \cdot \delta \mathbf{r}_i \right] = \delta \sum_i \frac{1}{2} m_i v_i^2 + \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i.$$

It is quite obvious that $\sum_i \frac{1}{2} m_i v_i^2$ is the kinetic energy, T , of the system of particles and if the forces are conservative, $\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = -\delta V$, where V is the potential energy. Thus right side of above equation is $\delta(T - V)$.

Hence

$$\frac{d}{dt} \left[\sum_i \left(m_i \frac{d\mathbf{r}_i}{dt} \right) \cdot \delta \mathbf{r}_i \right] = \delta(T - V).$$

Integrating above equation with respect to time from initial instant t_1 to final instant t_2 of the motion, we get

$$\left| \sum_i m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \mathbf{r}_i \right|_{t_1}^{t_2} = \int_{t_1}^{t_2} \delta(T - V) dt. \quad \dots (3)$$

But
$$\delta [(T - V) dt] = \delta (T - V) dt + (T - V) \delta (dt).$$

since
$$\delta (dt) = d(\delta t) = 0.$$

Thus
$$\int_{t_1}^{t_2} \delta (T - V) dt = \delta \int_{t_1}^{t_2} (T - V) dt.$$

Eq. (3) is therefore,

$$\left| \sum_i m_i \frac{d\mathbf{r}_i}{dt} \cdot \delta \mathbf{r}_i \right|_{t_1}^{t_2} = \delta \int_{t_1}^{t_2} (T - V) dt.$$

To get Hamilton's principle, we consider the motion of the system. In δ -variation process we account for all paths whether dynamical or varied with a condition that all paths have the same termini, i.e. no co-ordinate variation at end points of motion. It means $\delta \mathbf{r}_i = 0$ at the two ends of the path for every particle. All paths are also traversed in the same time. Thus left hand side of above equation is zero and we get

$$\delta \int_{t_1}^{t_2} (T - V) dt = 0,$$

which is Hamilton's principle. Putting $L = T - V$, we get the form

$$\delta \int_{t_1}^{t_2} L dt = 0.$$

2.7-2. DEDUCTION OF NEWTON'S SECOND LAW OF MOTION FROM HAMILTON'S PRINCIPLE

Suppose a particle of mass, m at the position $\mathbf{r} = \mathbf{r}(x, y, z)$ is moving under the action of a field of force $\mathbf{F}$. The kinetic energy of the particle is

$$T = \frac{1}{2} m |\dot{\mathbf{r}}|^2$$

and the variation of work done is

$$\delta W = \mathbf{F} \cdot \delta \mathbf{r} = -\delta V$$

Now from Hamilton's variational principle

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} (T - V) dt = \int_{t_1}^{t_2} (\delta T - \delta V) dt \\ &= \int_{t_1}^{t_2} (m\dot{\mathbf{r}} \cdot \delta \dot{\mathbf{r}} + \mathbf{F} \cdot \delta \mathbf{r}) dt \\ &= \int_{t_1}^{t_2} m\dot{\mathbf{r}} \cdot \delta \dot{\mathbf{r}} dt + \int_{t_1}^{t_2} \mathbf{F} \cdot \delta \mathbf{r} dt \end{aligned}$$

Integrating the first term on RHS by parts, we get

$$\begin{aligned} 0 &= m\dot{\mathbf{r}} \cdot \delta \mathbf{r} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{\mathbf{r}} \cdot \delta \mathbf{r} dt + \int_{t_1}^{t_2} \mathbf{F} \cdot \delta \mathbf{r} dt \\ &= - \int_{t_1}^{t_2} (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \delta \mathbf{r} dt \end{aligned}$$

because $\delta \mathbf{r}$ vanishes at end points. The above equation is true for every virtual displacement $\delta \mathbf{r}$ and hence the integrand must vanish. That is,

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

which is the Newton's second law of motion.

2.8. DEDUCTION OF LAGRANGE'S EQUATIONS USING VARIATIONAL PRINCIPLE FOR SYSTEMS INVOLVING FORCES NOT DERIVABLE FROM POTENTIAL FUNCTIONS

Non-conservative systems :

To include non-conservative forces, let us extend the principle to assume a form

$$\delta I = \delta \int_{t_1}^{t_2} (T + W) dt = 0, \quad \dots (1)$$

with fixed end points, where $\delta W = \delta \sum \mathbf{F}_i \cdot \mathbf{r}_i = \sum \mathbf{F}_i \cdot \delta \mathbf{r}_i$ represents the work done by the force on the system during the virtual displacement from actual to the rest conceived or rather varied paths. We shall explain it a bit more. It has been stated at several places in preceding articles that δ variation is a process that does not include time variation. Thus, the time of motion for the system along every path, whether actual or varied, is the same. It is possible only when we imagine the varied paths in configuration space to be built up by a succession of virtual displacements (as these do not involve a change of time) from the actual path A of motion. Thus δq_j refers to virtual displacement of the co-ordinates. Concept of generalised force can then also be applied here as indicated in D' Alembert's principle.

Possible paths are referred to as $q_j(t, \alpha)$. The transformation equations can be written as

$$\mathbf{r}_i = \mathbf{r}_i[q_j(t, \alpha), t],$$

from which we find that

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

Further, the components of generalised force are expressed as

$$\begin{aligned} \delta W &= \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i \\ &= \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\ &= \sum_j Q_j \delta q_j, \end{aligned}$$

because

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Thus eq. (1) takes the form

$$\delta \int_1^2 T dt + \int_1^2 \sum_j Q_j \delta q_j dt = 0 \quad \dots (2)$$

Further, since T is a function of q_j and $\dot{q}_j$, we write

$$\delta \int_1^2 T(q_j, \dot{q}_j) dt = \int_1^2 \sum_j \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt$$

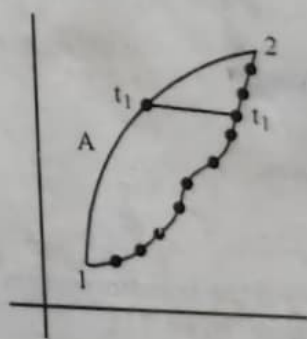


Fig. 2.4. Construction of varied paths in configuration space by virtual displacements from actual path.

$$= \int_1^2 \sum_j \left(\frac{\partial T}{\partial q_j} \delta q_j \right) dt + \sum_j \frac{\partial T}{\partial \dot{q}_j} \delta q_j \Big|_1^2 - \int_1^2 \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j dt.$$

The middle term is zero as it is a variation with fixed end point ; so R.H.S. is

$$\int_1^2 \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \right] \delta q_j dt. \quad \dots (3)$$

Eq. (3) is put in eq. (2) to get

$$\int_1^2 \sum_j \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j dt = 0.$$

Again, since the constraints are assumed to be holonomic, δq_j are independent of each other, and above integral can vanish if and only if the coefficients of δq_j separately vanish, i.e.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j,$$

which are Lagrange's equations of motion for *non-conservative* system.

2.9. APPLICATIONS OF LAGRANGE'S EQUATIONS OF MOTION

First we shall show that if the equations of transformation do not involve time then kinetic energy is a homogeneous quadratic function of velocities.

Kinetic energy T is given, in general, by

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j \right) \cdot \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right)$$

because from transformation equations

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n),$$

giving

$$\dot{\mathbf{r}}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j$$

Thus

$$\begin{aligned} T &= \sum_j \sum_k \left(\sum_i \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k \\ &= \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k, \end{aligned} \quad \dots (1)$$

only if the transformation equations do not contain time explicitly, i.e., constraints are independent of time. In eq. (1),

$$\begin{aligned} a_{jk} &= a_{jk}(q_1, \dots, q_n) = \sum_i \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \\ &= \sum_i \frac{1}{2} m_i \left(\frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k} + \frac{\partial y_i}{\partial q_j} \frac{\partial y_i}{\partial q_k} + \frac{\partial z_i}{\partial q_j} \frac{\partial z_i}{\partial q_k} \right). \end{aligned}$$

Eq. (1) explains that, for *holonomic scleronomous* system, kinetic energy is always a homogeneous quadratic form in generalised velocities, i.e., in $\dot{q}_j$.

When equations of transformation involve time :

Then

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_j, \dots, t),$$

giving

$$\frac{d\mathbf{r}_i}{dt} = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t},$$

so that the expression of kinetic energy now becomes

$$\begin{aligned}
 T &= \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \\
 &= \sum_{i,j} m_i \left(\sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right)^2 \\
 &= a + \sum_j a_j \dot{q}_j + \sum_{j,k} a_{jk} \dot{q}_j \dot{q}_k,
 \end{aligned}$$

where a, a_j, a_{jk} are definite functions of $\mathbf{r}$'s and t , and hence of the q 's and t . On comparing the last two steps of above equation, we can see that

$$a = \sum_i \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t} \right)^2,$$

$$a_j = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

$$a_{jk} = \sum_i \frac{1}{2} m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

and

Expression for T is now not a homogeneous quadratic in generalised velocities, but in addition it also includes first two terms. However, if transformation equations do not involve time explicitly i.e., constraints are independent of time then first two terms in T will vanish, leaving

$$T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k,$$

expressing that T is always a homogeneous quadratic form in generalised velocities.

To solve problems of mechanics by Lagrangian method, one is to set up, first, Lagrangian L which involves two scalars T and V . Then this value of L is substituted in the famous Lagrange's equations of motion. Transformation equations are used to transform T and V into generalised co-ordinates from cartesian co-ordinates.

2.9-1. LINEAR HARMONIC OSCILLATOR :

✓ Lagrange's equation of motion for one dimension, say x , can be written as :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0. \quad \dots (1)$$

The kinetic energy of this system is

$$T = \frac{1}{2} m \dot{x}^2$$

and potential energy is

$$\begin{aligned}
 V &= - \int \mathbf{F} \cdot d\mathbf{x} \\
 &= - \int -kx \, dx = \frac{1}{2} kx^2 + c
 \end{aligned}$$

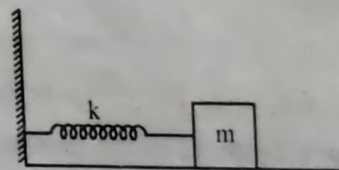


Fig. 2.5. Linear harmonic oscillator.

where c is a constant of integration and k is spring constant. If we choose the horizontal plane passing through the position of equilibrium as the reference level, then $V = 0$ at $x = 0$ so that $c = 0$. Thus Lagrangian is

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2,$$

so that

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and

$$\frac{\partial L}{\partial x} = -kx.$$

Eq. (1) takes the form

$$\frac{d}{dt} (m\dot{x}) - (-kx) = 0$$

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or $m\ddot{x} + kx = 0$,
 which is the required expression. It is an equation of simple harmonic motion and can be put in the form $\ddot{x} + \omega^2 x = 0$, where ω is the frequency of oscillation given by $\sqrt{k/m}$.

2.9-2. SIMPLE PENDULUM :

Refer to fig. (2.6). The angle θ between rest position and deflected position is chosen as generalized co-ordinate. If the string is of length l , then kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} mv^2 = \frac{1}{2} m (l \dot{\theta})^2 \\ &= \frac{1}{2} ml^2 \dot{\theta}^2, \end{aligned}$$

where m is the mass of the bob.

In coming from position B to A, the mass has fallen freely through a vertical distance CA. Thus potential energy is

$$\begin{aligned} V &= mg(OA - OC) \\ &= mg(l - l \cos \theta) \\ &= mgl(1 - \cos \theta), \end{aligned}$$

where the reference level or zero level of potential energy has been taken at a distance l below the point of suspension.

Thus Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos \theta), \end{aligned}$$

so that

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

and

$$\frac{\partial L}{\partial \theta} = -mgl \sin \theta.$$

Putting in Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0,$$

we get

$$\frac{d}{dt} (ml^2 \dot{\theta}) + mgl \sin \theta = 0,$$

or

$$ml^2 \ddot{\theta} + mgl \sin \theta = 0,$$

or

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0.$$

If the amplitude of the motion is small enough, that is, $\sin \theta = \theta$, then

$$\ddot{\theta} + \frac{g}{l} \theta = 0,$$

which is equation for simple harmonic motion with period

$$T = 2\pi \sqrt{\left(\frac{l}{g} \right)}.$$

2.9-3. SPHERICAL PENDULUM

In a simple pendulum, bob moves on a circle but a pendulum in which bob is constrained to move on a sphere rather than on a circle is called spherical pendulum. The position of the bob is located by spherical co-ordinate θ, ϕ as shown in fig. (2.7). The distance r of the bob from the origin will be the radius of the sphere on which the particle (or bob) moves. The force in this case will not be central, but is constant in the vertical direction.

The kinetic energy of the bob

$$= \frac{1}{2} mr^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

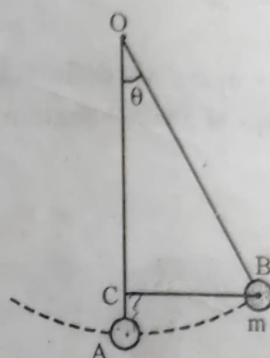


Fig. 2.6. Simple pendulum.

The potential energy of the bob due to gravity relative to the horizontal plane

$$= mgr \cos \theta.$$

The Lagrangian for the system is thus given by

$$L = T - V$$

$$= \frac{1}{2} mr^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - mgr \cos \theta. \quad \dots (1)$$

From above expression for L it follows that

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta},$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin \theta \cos \theta \dot{\phi} + mgr \sin \theta$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \dot{\phi}$$

$$\frac{\partial L}{\partial \phi} = 0.$$

and

... (2)

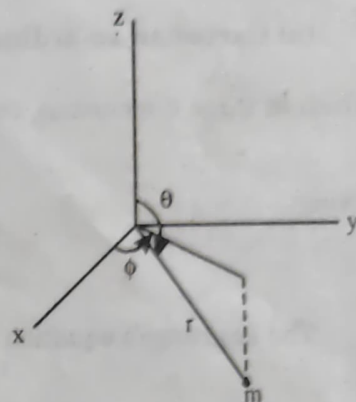


Fig. 2.7.

The Lagrange's equations for such a system can be written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0$$

which, on applying eq. (2), becomes

$$\frac{d}{dt} (mr^2 \dot{\theta}) - mr^2 \sin \theta \cos \theta \dot{\phi}^2 - mgr \sin \theta = 0$$

and

$$\frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) = 0.$$

On solving further, we get

$$\left. \begin{aligned} mr^2 \ddot{\theta} - mr^2 \sin \theta \cos \theta \dot{\phi}^2 - mgr \sin \theta &= 0 \\ mr^2 \frac{d}{dt} (\sin^2 \theta \dot{\phi}) &= 0 \end{aligned} \right\}$$

and

which are the desired equations of the motion of a spherical pendulum.

2.9-4. ISOTROPIC OSCILLATOR (Three dimensional).

An isotropic harmonic oscillator can be considered as a vibrating particle that is acted upon by a force directed always towards or away from the position of equilibrium and the magnitude of which varies linearly with the distance from the position of equilibrium. Such a force can be represented as $F = -kr$ where k is called the force constant.

The potential energy is

$$\begin{aligned} V &= - \int F dr + c \\ &= - \int -kr dr + c = \frac{1}{2} kr^2 + c \end{aligned}$$

If we choose the horizontal plane passing through the position of equilibrium as the reference level, then $V = 0$ at $r = 0$, making the constant, c , zero. Thus

$$V = \frac{1}{2} kr^2.$$

We shall develop Lagrangian equations of motion for a three dimensional isotropic oscillator in cartesian and polar co-ordinates.

(a) **Cartesian co-ordinates** : The Lagrangian L for the oscillator can be written as

$$L = T - V,$$

which in three dimensions, on using cartesian co-ordinates, takes the form

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} k (x^2 + y^2 + z^2), \quad \dots (1)$$

giving

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x}, & \frac{\partial L}{\partial \dot{y}} &= m\dot{y}, & \frac{\partial L}{\partial \dot{z}} &= m\dot{z}, \\ \frac{\partial L}{\partial x} &= -kx, & \frac{\partial L}{\partial y} &= -ky, & \frac{\partial L}{\partial z} &= -kz. \end{aligned} \right\} \quad \dots (2)$$

The Lagrange's equation in cartesian co-ordinates are

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} &= 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} &= 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} &= 0, \end{aligned} \right\}$$

or

$$\left. \begin{aligned} \frac{d}{dt} (m\dot{x}) + kx &= m\ddot{x} + kx = 0 \\ m\ddot{y} + ky &= 0 \\ m\ddot{z} + kz &= 0. \end{aligned} \right\} \quad \dots (3)$$

Eq. (3) gives the desired equations of motion for three dimensional isotropic oscillator.

(b) **Polar co-ordinates** : Let the polar co-ordinates of the particle at any instant be (r, θ, ϕ) . If the cartesian co-ordinates of particle at the instant be (x, y, z) , then the transformation equations will be

$$\left. \begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta. \end{aligned} \right\} \quad \dots (4)$$

The kinetic energy of the particle is

$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \end{aligned}$$

and the potential energy will be

$$V = \frac{1}{2} kr^2.$$

The Lagrangian will, therefore, be

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - \frac{1}{2} kr^2 \end{aligned}$$

giving

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\ \frac{\partial L}{\partial \dot{\theta}} &= mr\dot{\theta} \\ \frac{\partial L}{\partial \dot{\phi}} &= mr^2 \sin^2 \theta \dot{\phi} \\ \frac{\partial L}{\partial r} &= m\dot{\theta}^2 + m\dot{\phi}^2 \sin^2 \theta - kr \\ \frac{\partial L}{\partial \theta} &= 2mr\dot{\phi}^2 \sin \theta \cos \theta \\ \frac{\partial L}{\partial \phi} &= 0. \end{aligned} \right\} \quad \dots (5)$$

The Lagrange's equations in polar co-ordinates are

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} &= 0. \end{aligned} \right\} \dots (6)$$

On applying eqs. (5) to eqs. (6) we get

$$\left. \begin{aligned} \frac{d}{dt} (m\dot{r}) - m\dot{\theta}^2 - m\dot{\phi}^2 \sin^2 \theta + kr &= 0 \\ \text{or } m\ddot{r} - m\dot{\theta}^2 - m\dot{\phi}^2 \sin^2 \theta + kr &= 0 \\ \text{and } \frac{d}{dt} (mr^2\dot{\theta}) - mr^2 \sin \theta \cos \theta \dot{\phi}^2 &= 0 \\ \text{and } \frac{d}{dt} (mr^2 \sin \theta \dot{\phi}) &= 0 \end{aligned} \right\} \dots (7)$$

Eqs. (7) are the required equations of motion in polar co-ordinates.

2.9-5. DUMB BELL

We shall set up the equations of motion of a dumb bell in a vertical plane. A system, consisting of two particles of masses m_1 and m_2 that are rigidly fastened to a rod of length l and negligible mass, is termed a dumb bell. The masses are free to move in the vertical plane XY under the action of gravity.

Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the centres of two masses m_1 and m_2 respectively. If θ is the inclination of the rod of length l we find with X axis,

$$\left. \begin{aligned} x_2 &= x_1 + l \cos \theta \\ y_2 &= y_1 + l \sin \theta \end{aligned} \right\} \dots (1)$$

so that

$$\left. \begin{aligned} \dot{x}_2 &= \dot{x}_1 - l \sin \theta \dot{\theta} \\ \dot{y}_2 &= \dot{y}_1 + l \cos \theta \dot{\theta} \end{aligned} \right\} \dots (2)$$

The kinetic energy of system is

$$\begin{aligned} T &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 [(\dot{x}_1 - l \sin \theta \dot{\theta})^2 + (\dot{y}_1 + l \cos \theta \dot{\theta})^2] \\ &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 [\dot{x}_1^2 + l^2 \sin^2 \theta \dot{\theta}^2 - 2\dot{x}_1 l \sin \theta \dot{\theta} \\ &\quad + \dot{y}_1^2 + l^2 \cos^2 \theta \dot{\theta}^2 + 2\dot{y}_1 l \cos \theta \dot{\theta}] \\ &= \frac{1}{2} (m_1 + m_2) (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 [l^2 \dot{\theta}^2 - 2l\dot{x}_1 \dot{\theta} \sin \theta + 2l\dot{y}_1 \dot{\theta} \cos \theta]. \end{aligned} \dots (3)$$

Taking X -axis as the reference level, the potential energy of the system is given by

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= m_1 g y_1 + m_2 g (y_1 + l \sin \theta) \\ &= (m_1 + m_2) g y_1 + m_2 g l \sin \theta. \end{aligned} \dots (4)$$

The Lagrangian L is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} (m_1 + m_2) (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 [l^2 \dot{\theta}^2 - 2l\dot{x}_1 \dot{\theta} \sin \theta \\ &\quad + 2l\dot{y}_1 \dot{\theta} \cos \theta] - (m_1 + m_2) g y_1 - m_2 g l \sin \theta. \end{aligned} \dots (5)$$

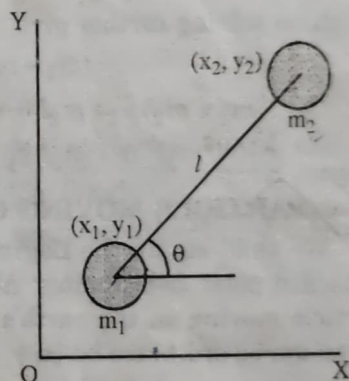


Fig. 2.8.

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{x}_1} &= (m_1 + m_2) \dot{x}_1 - m_2 l \dot{\theta} \sin \theta, & \frac{\partial L}{\partial \dot{y}_1} &= (m_1 + m_2) \dot{y}_1 + m_2 l \dot{\theta} \cos \theta \\ \frac{\partial L}{\partial x_1} &= 0, & \frac{\partial L}{\partial y_1} &= -(m_1 + m_2) g \\ & & \frac{\partial L}{\partial \dot{\theta}} &= m_2 l^2 \ddot{\theta} - m_2 l \dot{x}_1 \sin \theta + m_2 l \dot{y}_1 \cos \theta \\ & & \frac{\partial L}{\partial \theta} &= -m_2 g l \cos \theta - m_2 l \dot{\theta} (\dot{y}_1 \sin \theta + \dot{x}_1 \cos \theta) \end{aligned} \right\} \dots (6)$$

Thus Lagrange's equations of motion will be

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} &= \frac{d}{dt} [(m_1 + m_2) \dot{x}_1 - m_2 l \dot{\theta} \sin \theta] = 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}_1} \right) - \frac{\partial L}{\partial y_1} &= \frac{d}{dt} [(m_1 + m_2) \dot{y}_1 + m_2 l \dot{\theta} \cos \theta] + (m_1 + m_2) g = 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= \frac{d}{dt} [m_2 l^2 \ddot{\theta} - m_2 l \dot{x}_1 \sin \theta + m_2 l \dot{y}_1 \cos \theta] + m_2 g l \cos \theta \\ &\quad + m_2 l \dot{\theta} (\dot{y}_1 \sin \theta + \dot{x}_1 \cos \theta) \end{aligned} \right\} \dots (7)$$

which on solving further give

$$\left. \begin{aligned} (m_1 + m_2) \ddot{x}_1 - m_2 l \ddot{\theta} \sin \theta - m_2 l \dot{\theta}^2 \cos \theta &= 0 \\ (m_1 + m_2) \ddot{y}_1 + m_2 l \ddot{\theta} \cos \theta - m_2 l \dot{\theta}^2 \sin \theta + (m_1 + m_2) g &= 0 \\ m_2 l^2 \ddot{\theta} - m_2 l \dot{x}_1 \sin \theta + m_2 l \dot{y}_1 \cos \theta + m_2 g l \cos \theta &= 0. \end{aligned} \right\} \dots (8)$$

2.9-6. PARTICLE MOVING ON THE SURFACE OF EARTH

We shall set up the Lagrange's equations of a particle moving on the surface of earth using spherical polar co-ordinates. At any instant, let (r, θ, ϕ) be the spherical polar co-ordinates of the particle moving on the earth's surface taking centre of the earth as origin. If the corresponding cartesian co-ordinates be (x, y, z) then transformation equations will be

$$\left. \begin{aligned} x &= r \sin \theta \cos \phi, \\ y &= r \sin \theta \sin \phi, \\ z &= r \cos \theta. \end{aligned} \right\} \dots (1)$$

giving

$$\left. \begin{aligned} \dot{x} &= r \cos \phi \cos \theta \dot{\theta} - r \sin \phi \sin \theta \dot{\phi}, \\ \dot{y} &= r \cos \phi \sin \theta \dot{\phi} + r \sin \phi \cos \theta \dot{\theta}, \\ \dot{z} &= -r \sin \theta \dot{\theta}. \end{aligned} \right\}$$

Differentiation of $\dot{r}$ is zero, since it is constant as the particle is moving on the surface of earth. If mass of the particle be m , its kinetic energy will be

$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= m [(r \cos \phi \cos \theta \dot{\theta} - r \sin \phi \sin \theta \dot{\phi})^2 \\ &\quad + (r \cos \phi \sin \theta \dot{\phi} + r \sin \phi \cos \theta \dot{\theta})^2 + (-r \sin \theta \dot{\theta})^2] \\ &= \frac{1}{2} m r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2). \end{aligned}$$

The potential energy will be a function of the particle's co-ordinates and can be expressed as $V(r, \theta, \phi)$. Therefore, the Lagrangian will be

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - V(r, \theta, \phi). \end{aligned} \dots (2)$$

The Lagrangian equations of motion for such a case will be

or

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} &= 0 \end{aligned} \right\} \quad \dots (3)$$

since θ and ϕ are the only generalised co-ordinates while r is constant.

From eq. (2), we find that

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= mr^2 \dot{\theta}, & \frac{\partial L}{\partial \dot{\phi}} &= mr^2 \sin^2 \theta \dot{\phi} \\ \frac{\partial L}{\partial \theta} &= mr^2 \sin \theta \cos \theta \dot{\phi}^2 - \frac{\partial V}{\partial \theta}, & \frac{\partial L}{\partial \phi} &= -\frac{\partial V}{\partial \phi} \end{aligned} \right\} \quad \dots (4)$$

Putting these eq. (4) in eq. (3), we get

$$\left. \begin{aligned} \frac{d}{dt} (mr^2 \dot{\theta}) - mr^2 \sin \theta \cos \theta \dot{\phi}^2 + \frac{\partial V}{\partial \theta} &= 0 \\ \frac{d}{dt} (mr^2 \sin^2 \theta \dot{\phi}) + \frac{\partial V}{\partial \phi} &= 0 \end{aligned} \right\}$$

or

$$\left. \begin{aligned} mr^2 \ddot{\theta} - mr^2 \sin \theta \cos \theta \dot{\phi}^2 + \frac{\partial V}{\partial \theta} &= 0 \\ mr^2 \sin^3 \theta \dot{\phi} + 2mr^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + \frac{\partial V}{\partial \phi} &= 0 \end{aligned} \right\} \quad \dots (5)$$

Eqs. (5) are desired equations of motion.

2.9-7. PARTICLE MOVING UNDER A CENTRAL FORCE

Central force is that force which is always directed away or towards a fixed centre and the magnitude of which is a function of the distance from the fixed centre (origin).

Let us consider a particle of mass m moving in a plane and attracted towards the origin of co-ordinates with a force that is inversely proportional to the square of the distance from it. If (r, θ) be its plane polar co-ordinates, the kinetic energy will be

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2).$$

and the potential energy

$$V = - \int_{\infty}^r F dr = - \int_{\infty}^r -\frac{k}{r^2} dr = -\frac{k}{r},$$

because force F is attractive in nature and varies as the square of the distance from the origin, i.e., $F = -k/r^2$, where k is a constant of proportionality. The Lagrangian will, thus, become

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r} \end{aligned} \quad \dots (1)$$

giving

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r}, & \frac{\partial L}{\partial \dot{\theta}} &= mr^2 \dot{\theta} \\ \frac{\partial L}{\partial r} &= m\dot{\theta}^2 - \frac{k}{r^2}, & \frac{\partial L}{\partial \theta} &= 0. \end{aligned} \right\} \quad \dots (2)$$

The equations of motion for such a case will be

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} &= 0 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= 0 \end{aligned} \right\} \quad \dots (3)$$

On applying eqs. (2) to eqs. (3) we find that

$$\left. \begin{aligned} \frac{d}{dt}(mr\dot{r}) - mr\dot{\theta}^2 + \frac{k}{r^2} &= 0 \\ \text{and} \quad \frac{d}{dt}(mr^2\dot{\theta}) &= 0 \end{aligned} \right\}$$

Solving further, we get

$$\left. \begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} &= 0 \\ \text{and} \quad 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} &= 0 \end{aligned} \right\} \quad \dots (4)$$

Eqs. (4) are the desired equations of motion for the particle moving under the influence of a central force.

2.9-8. ELECTRICAL CIRCUIT

The Lagrangian for an electrical circuit consisting of finite number of inductances, capacitances, and resistances is expressed as

$$L_E = T_M - V_E$$

where L_E is the Lagrangian for the electrical circuit, T_M is the magnetic energy of the electrical circuit (analogous to the kinetic energy of the mechanical system) and V_E is the electrical energy of the electrical circuit (analogous to the potential energy of the mechanical system).

The Lagrange's equations of motion for such a circuit can be written as

$$\frac{d}{dt} \left(\frac{\partial L_E}{\partial \dot{q}_j} \right) - \frac{\partial L_E}{\partial q_j} = Q_j$$

where Q_j is the generalised force due to friction, while the conservative force are included in L_E . If the system is free from any frictional or dissipative forces, $Q_j = 0$, so that equations of motion are

$$\frac{d}{dt} \left(\frac{\partial L_E}{\partial \dot{q}_j} \right) - \frac{\partial L_E}{\partial q_j} = 0.$$

● **Ex. 1.** To find the Lagrange's equations of motion for an electrical circuit comprising an inductance L and capacitance C . The condenser is charged to q coulombs and the current flowing in the circuit is i amperes.

For such a circuit the magnetic energy will be $T_M = \frac{1}{2} Li^2 = \frac{1}{2} L\dot{q}^2$ and electrical energy $V_E = \frac{1}{2} q^2/C$.

Therefore, Lagrangian can be represented as

$$L_E = \frac{1}{2} L\dot{q}^2 - \frac{1}{2} q^2/C. \quad \dots (1)$$

Choosing q as the generalised co-ordinate, the equation of motion will be

$$\frac{d}{dt} \left(\frac{\partial L_E}{\partial \dot{q}} \right) - \frac{\partial L_E}{\partial q} = 0 \quad \dots (2)$$

But from eq. (1),

$$\frac{\partial L_E}{\partial \dot{q}} = L\dot{q} \quad \text{and} \quad \frac{\partial L_E}{\partial q} = -\frac{q}{C}$$

so the eq. (2) takes the form

$$L\ddot{q} + \frac{q}{C} = 0, \quad \text{or} \quad \ddot{q} + \frac{1}{LC} q = 0, \quad \text{where } f = \frac{1}{2\pi\sqrt{LC}}, \quad T = 2\pi\sqrt{LC}$$

or

$$3mg \cos \theta - 2mg \geq 0$$

For equality, $\cos \theta_c = 2/3$ where θ_c is the angle at which the particle flies off the surface.

2.11. CONSERVATION THEOREMS

If any system or any function representing a property of the system does not change under some operation, the system is said to possess a *symmetry* with respect to the given operation. For example, when a cylinder is rotated about its axis by an arbitrary angle, its apparent shape does not change. The cylinder is said to have rotational symmetry about its axis. Co-ordinate which corresponds to such an operation becomes cyclic and operation has no effect on the description of the system-motion. Such symmetry operations under arbitrary translation give rise to conservation of linear momentum, and under arbitrary rotation about arbitrary axis give rise to conservation of angular momentum. Thus symmetries correspond to conservation laws.

The study of conservation theorems for a system in motion provides the constants of motion, the knowledge of which helps in describing the motion of the system. The Lagrange's equations of motion are very appropriate for recognising the constants of motion as we shall see in this article.

Lagrange's equations of motion are second order in time. To solve them completely, for a system of n degrees of freedom, $2n$ constants of integration will be involved which can be determined if n initial values for q_j and n initial values for $\dot{q}_j$ are known. It is not always possible to integrate every equation of motion so as to get a complete solution in terms of known functions. But often in such cases it is possible to extract sufficient information about the physical nature of the motion of the system, without solving the problem completely, through the constants of motion. The conservation theorems to be discussed in this article are *first integrals of equations of motion*. These are the relations of the type $f(q_j, \dot{q}_j, t) = \text{constant}$, and are first order differential equations.

Cyclic or Ignorable Co-ordinate : We have shown that Lagrangian L is a function of generalised co-ordinate q_j , generalised velocity $\dot{q}_j$ and time t . Now if the Lagrangian of a system

does not contain a particular co-ordinate q_k , then obviously for such a system $\frac{\partial L}{\partial q_k} = 0$. Such a

co-ordinate is referred to as an *ignorable* or *cyclic co-ordinate*. We shall explain in next sub-heading* how the co-ordinate symmetry operations, which leave the description of motion invariant, determine the cyclic co-ordinate.

Generalised momentum : It is also termed as conjugate or canonical momentum.

Let us consider a system of mass points acted upon by forces derived from potentials dependent on position only. For such a system, called conservative one, Lagrange's equations of motion are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0.$$

Suppose q_j co-ordinate is cyclic i.e., it does not occur in Lagrangian L , then for this co-ordinate Lagrange's equation reduces to

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0,$$

which on integration yields

$$\frac{\partial L}{\partial \dot{q}_j} = \text{constant}.$$

It is quite easy to recognise $\partial L / \partial \dot{q}_j$ as generalised momentum if we look at the following expression :

* As we shall see in generalised momentum that the first integral of Lagrange's equation of motion gives

$$\frac{\partial L}{\partial \dot{q}_j} = p_j = \text{constant}.$$

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \Sigma \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = m_i \dot{x}_i = p_{ix},$$

which is x-component of the linear momentum associated with the i^{th} particle. Generalising the concept, a momentum associated with the co-ordinate q_j shall be

$$p_j = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}$$

and is termed as generalised momentum. Thus

$$p_j = \text{constant}$$

which means that *the generalised momentum conjugate to a cyclic co-ordinate is conserved.*

If we put p_j in Lagrange's equation, we get

$$\dot{p}_j = \frac{\partial L}{\partial q_j}$$

(1) Conservation of linear momentum : We shall show that if a co-ordinate corresponding to a displacement is cyclic, translation of the system has no effect on the problem, i.e., description of system motion remains invariant under such a translation and linear momentum is conserved.

Let us consider first q_j as the generalised co-ordinate and dq_j as representing the translation of the system as a whole in some given direction. Obviously q_j will not appear in T because it depends upon velocities which involve time derivative of displacement and hence, in turn, are unaffected by a shift in origin. Therefore $\frac{\partial T}{\partial q_j}$ is zero. Further, we consider a conservative system so

that $\frac{\partial V}{\partial \dot{q}_j} = 0$ and only $\frac{\partial V}{\partial q_j}$ exists. Then Lagrange equation of motion for such a co-ordinate is defined as

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + \frac{\partial V}{\partial q_j} = 0$$

or

$$\dot{p}_j = - \frac{\partial V}{\partial q_j} = Q_j \quad \dots (1)$$

Now if we show that Q_j represents the component of total force along the direction of translation of q_j and p_j is the component of total linear momentum along the same direction, then equation (1) will represent the equation of motion for total linear momentum.

Generalised force is given by

$$Q_j = \Sigma_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

If $\mathbf{n}$ is the unit vector along the direction of translation, then

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n}$$

or

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n},$$

Thus

$$Q_j = \Sigma_i \mathbf{F}_i \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{F},$$

which represents the component of total force along the direction of $\mathbf{n}$ i.e. along the direction of translation of q_j .

Further kinetic energy is

$$T = \frac{1}{2} \Sigma_i m_i \dot{\mathbf{r}}_i^2,$$

so that generalised momentum is

$$\begin{aligned}
 p_j &= \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} \\
 &= \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \\
 &= \sum_i m_i \mathbf{v}_i \cdot \mathbf{n} \\
 &= \mathbf{n} \cdot \sum_i m_i \mathbf{v}_i,
 \end{aligned}$$

which shows that p_j represents the component of total linear momentum along the direction of translation.

Now we can say that equation

$$\dot{p}_j = Q_j$$

is the equation of motion for total linear momentum of the system.

Now suppose q_j is a cyclic co-ordinate, then $-\frac{\partial V}{\partial q_j} = Q_j = 0$.

This gives

$$\dot{p}_j = 0$$

or

$$p_j = \text{constant}.$$

Thus if a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.

Since q_j is a cyclic co-ordinate, it would not appear in Lagrangian $L(q_j, \dot{q}_j, t)$. Thus L will remain unaffected by the translation dq_j of the system so that equations of motion remain invariant or rather, description of the motion of the system is left invariant under such a co-ordinate translation and linear momentum is conserved.

(2) Conservation of angular momentum : We shall show that if a co-ordinate corresponding to a rotation is cyclic, rotation of the system about the given axis has no effect on the description of the system motion i.e. system remains invariant under such a co-ordinate rotation and angular momentum is conserved.

If q_j be first assumed as generalised co-ordinate, then since T does not depend upon q_j and V does not depend, for a conservative system, on $\dot{q}_j$, we get

$$\dot{p}_j = Q_j. \quad \dots (2)$$

If we show that with q_j , a rotation co-ordinate, the generalised force is the component of the total applied torque about the axis of rotation and p_j is the component of total angular momentum along the same axis, equation (2) will represent the equation of motion for the total angular momentum.

The generalised force Q_j is given by

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

Here change in q_j must correspond to an infinitesimal rotation of the vector $\mathbf{r}_i$. Keeping the magnitude of vector constant, we find

$$\begin{aligned}
 |d\mathbf{r}_i| &= AB dq_j \\
 &= r_i \sin \theta dq_j
 \end{aligned}$$

so that

$$\left| \frac{\partial \mathbf{r}_i}{\partial q_j} \right| = r_i \sin \theta,$$

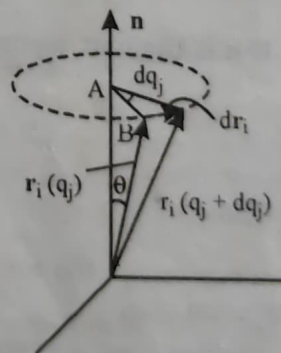


Fig. 2.14. Change of position vector under rotation of the system.

the direction of which is perpendicular to both $\mathbf{r}_i$ and $\mathbf{n}$.

Thus

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \mathbf{n} \times \mathbf{r}_i.$$

Generalised force can be expressed as

$$\begin{aligned} Q_j &= \sum_i \mathbf{F}_i \cdot \mathbf{n} \times \mathbf{r}_i \\ &= \sum_i \mathbf{F}_i \cdot (\mathbf{n} \times \mathbf{r}_i) \\ &= \sum_i (\mathbf{n} \times \mathbf{r}_i) \cdot \mathbf{F}_i \\ &= \sum_i \mathbf{n} \cdot \mathbf{r}_i \times \mathbf{F}_i \\ &= \sum_i \mathbf{n} \cdot \mathbf{N}_i = \mathbf{n} \cdot \sum_i \mathbf{N}_i \\ &= \mathbf{n} \cdot \mathbf{N}, \end{aligned}$$

where $\mathbf{N}$ is the total torque. The expression shows that Q_j is the component of total torque about the axis of rotation. Further

$$\begin{aligned} p_j &= \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \\ &= \sum_i m_i \mathbf{v}_i \cdot \mathbf{n} \times \mathbf{r}_i \\ &= \sum_i \mathbf{n} \cdot \mathbf{r}_i \times m_i \mathbf{v}_i \\ &= \mathbf{n} \cdot \sum_i \mathbf{L}_i = \mathbf{n} \cdot \mathbf{L}, \end{aligned}$$

which shows that p_j is the component of total angular momentum along the axis of rotation and hence equation (2) represents equation of motion for the total angular momentum of the system.

Suppose q_j is a cyclic co-ordinate, then

$$Q_j = -\frac{\partial V}{\partial q_j} = 0 \quad \text{and hence } \dot{p}_j = 0$$

or

$$p_j = \text{constant}$$

or

$$\mathbf{n} \cdot \mathbf{L} = \text{constant},$$

i.e., if the rotation co-ordinate q_j is cyclic, then Q_j , which is the component of the applied torque along $\mathbf{n}$ vanishes and the component of $\mathbf{L}$ along $\mathbf{n}$ is constant.

Since q_j , rotation co-ordinate, is cyclic, L will remain unaffected under such a rotation and hence equations of motion will be invariant. This shows that the description of the system motion under such a rotation about a given axis is left invariant and the angular momentum is conserved. For *spherically symmetric problems*, e.g., motion of the system of particles under the action of a central force and internal forces, whose form is independent of the orientation of the co-ordinate system, the total angular momentum vector is a constant of motion. For example, refer to art. 2.9-7 about motion under central force. We have the Lagrangian

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r}.$$

Obviously, θ is cyclic co-ordinate in this case. We find from the resulting equation of motion that

$$\frac{d}{dt} (mr^2 \dot{\theta}) = 0$$

or

$$mr^2 \dot{\theta} = \text{constant},$$

which is angular momentum. Thus when θ is cyclic, corresponding angular momentum is conserved (note Q_θ is also zero).

Thus we conclude, in general, that if under a co-ordinate transformation, functional dependence of the Lagrangian on the new co-ordinates and new generalised velocities is the same as its dependence on the old co-ordinates and old generalised velocities, then the description of the motion of a system of particles is left invariant. Further *co-ordinate symmetry operations*, which result in the existence of cyclic co-ordinates, leave the description of the motion of the system invariant. Refer to art 2.15.

(3) Conservation of energy : Jacobi's integral : Let us consider :

(i) a *conservative system* so that potential energy is a function of co-ordinates only and not that of velocities,

(ii) *constraints do not change with time*, i.e. they are independent of time and consequently, equations of transformation* to generalised co-ordinates do not involve time explicitly and hence

(iii) L can be written as $L(q_j, \dot{q}_j)$.

Thus its total time derivative will be

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt}$$

Putting

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right)$$

from Lagrange's equation, we get

$$\begin{aligned} \frac{dL}{dt} &= \sum_j \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} \right] \\ &= \sum_j \frac{d}{dt} \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) \\ &= \sum_j \frac{d}{dt} \left(\dot{q}_j \frac{\partial T}{\partial \dot{q}_j} \right) \end{aligned}$$

because for a conservative system

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \quad \text{as} \quad \frac{\partial V}{\partial \dot{q}_j} = 0. \quad \text{Also} \quad p_j = \frac{\partial T}{\partial \dot{q}_j}$$

Thus

$$\frac{dL}{dt} - \sum_j \frac{d}{dt} (\dot{q}_j p_j) = 0$$

or

$$\frac{d}{dt} (\sum_j \dot{q}_j p_j - L) = 0$$

or

$$\sum_j \dot{q}_j p_j - L = \text{constant} = J.$$

The constant of motion, J , is one of the *first integrals of equations of motion* and is, by definition, called *Jacobi's integral of the system* and we will show that, with all requirements stated in the beginning, it represents the total energy of the system. If all q_j 's are substituted properly by functions of p_j 's then function J is nothing but Hamiltonian, H . So

$$H = \sum_j \dot{q}_j p_j - L = \text{constant} \quad \dots (3)$$

We have shown in Art. 2.9 that when constraints are independent of time, i.e., equations of transformation do not involve time explicitly and constraints are holonomic, then *kinetic energy* can be expressed as *homogeneous quadratic function of generalised velocities* and therefore

$$T = \sum_{jk} a_{jk} \dot{q}_j \dot{q}_k$$

* $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n)$.

Euler's theorem states that if f is a homogeneous function, of order n , of a set of variable q_j , then

$$\sum_j q_j \frac{\partial f}{\partial q_j} = nf.$$

But here n is 2 so that

$$\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T$$

or

$$\sum_j \dot{q}_j p_j = 2T$$

Thus from equation (3)

$$H = 2T - L = 2T - (T - V)$$

or

$$H = T + V$$

which shows that H equals the total energy and is conserved.

As t does not occur in L explicitly, it is cyclic. We have defined while interpreting conjugate momentum that the generalised momentum conjugate to a cyclic co-ordinate q_j is conserved; we can then interpret that *energy is essentially the generalised momentum conjugate to time*.

It may happen that H be a constant of motion (conserved) but *not* the total energy. Suppose constraints are independent of time even then co-ordinate transformation equations $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2 \dots q_n, t)$ may involve time explicitly because of rotation of co-ordinate axes or due to some other reasons; then we cannot write

$$\sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T,$$

since kinetic energy is no more a homogeneous quadratic function of velocities and thus H will not be $(T + V)$, the total energy. On the other hand, L still may not involve time explicitly so that the quantity $\frac{d}{dt} (\sum_j \dot{q}_j p_j - L) = 0$ or $\frac{dH}{dt} = 0$ or H is still conserved, i.e. it continues to be a constant of motion. *Therefore identification of H as a constant of motion and as the total energy are two separate matters and the conditions sufficient for one are not enough for other as stated above.*

Thus Lagrangian, L , need not always be defined as $(T - V)$ or $(T - U)$ and the existence of Jacobi's integral would not necessarily mean that the system is conservative. But for any conservative system, irrespective of its potential energy functions (V or U), its total energy is always given by $(T + V)$, never by $(T + U)$, because the velocity dependent part of the potential energy, U , does not enter into the expression for the total energy on account of the work done by the gyroscopic forces being zero.

2.12. PROCEDURE TO ELIMINATE CONSIDERATION OF IGNORABLE COORDINATES : THE ROUTHIAN FUNCTION

While dealing with Kepler's problem, we have shown that Lagrangian is given by

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r} \quad (\text{Art. 2.9-7 eq. 1})$$

where k is force constant. The particle is moving in a plane with (r, θ) as polar co-ordinates.

We note that co-ordinate θ does not occur in the Lagrangian, L and is therefore an ignorable co-ordinate so that corresponding momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = l = \text{a constant},$$

It appears from the expression for Lagrangian that we can proceed to solve the Kepler problem without considering the co-ordinate θ which is ignorable. But this is not so in Lagrangian formulation because we still have to care for θ which requires an information as to how θ varies

Putting eqs. (4) in Lagrange's equations

$$\sum_{j=1}^n \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] = 0,$$

we get the equations

$$\sum_{j=k+1}^n \left[\frac{d}{dt} \left(\frac{\partial R}{\partial \dot{q}_j} \right) - \frac{\partial R}{\partial q_j} \right] = 0 \quad \dots (5)$$

in which Routhian function has replaced the Lagrangian function. These are only $(n-k)$ second-order equations in the *non-ignorable* variables. That is how, through Routhian procedure, we can eliminate the ignorable co-ordinates from the equations of motion.

Kepler's Problem : Let us now apply Routhian procedure to the Kepler's problem. We know that

$$p_\theta = mr^2 \dot{\theta}$$

so that

$$\begin{aligned} R &= L - p_\theta \dot{\theta} = L - mr^2 \dot{\theta}^2 \\ &= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{k}{r} - mr^2 \dot{\theta}^2 \\ &= \frac{1}{2} m (\dot{r}^2 - r^2 \dot{\theta}^2) + \frac{k}{r} \end{aligned}$$

giving

$$\frac{\partial R}{\partial \dot{r}} = m\dot{r}$$

and

$$\frac{\partial R}{\partial r} = -mr\dot{\theta}^2 - \frac{k}{r^2}$$

Putting in eq. (5), we get the equation of motion as

$$\text{or} \quad \frac{d}{dt} (m\dot{r}) - mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

or

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r^2} = 0$$

or

$$\begin{aligned} m\ddot{r} &= mr\dot{\theta}^2 - \frac{k}{r^2} \\ &= \frac{mv_\theta^2}{r} + f(r) \end{aligned}$$

which is the equation of a particle moving under central force and is same as obtained in art. 2.9-7 using Lagrange's equation of motion.

2.13. GAUGE FUNCTION FOR LAGRANGIAN :

Euler-Lagrange's equations does not change if we add to the Lagrangian a *total* time derivative of any arbitrary function $F(q_j, t)$. Such a function is called *Gauge function* for Lagrangian. It means $\left[L(q_j, \dot{q}_j, t) + \frac{dF}{dt}(q_j, t) \right]$ also satisfies the Lagrange's equations of motion.

We shall show it. Function is

$$F = F(q_i, t)$$

then

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_i} \cdot \dot{q}_i + \frac{\partial F}{\partial t}$$

which when substituted in the equation of motion should yield zero so that differential of $\left(L + \frac{dF}{dt}\right)$ is same as L giving same set of Lagrange's equations. Therefore

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial (dF/dt)}{\partial \dot{q}_j} \right) - \frac{\partial}{\partial q_j} \left(\frac{dF}{dt} \right) &= \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\frac{\partial F}{\partial t} + \dot{q}_i \frac{\partial F}{\partial q_i} \right) \right] - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} + \dot{q}_i \frac{\partial F}{\partial q_i} \right) \\ &= \frac{d}{dt} \left[\frac{\partial}{\partial t} \left(\frac{\partial F}{\partial \dot{q}_j} \right) + \dot{q}_i \frac{\partial}{\partial q_i} \left(\frac{\partial F}{\partial \dot{q}_j} \right) + \frac{\partial F}{\partial q_i} \frac{\partial \dot{q}_i}{\partial \dot{q}_j} \right] \\ &\quad - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_i} \right) \dot{q}_i - \frac{\partial F}{\partial q_i} \frac{\partial \dot{q}_i}{\partial q_j} \end{aligned}$$

As $F \neq F(\dot{q}_j)$ so $\frac{\partial F}{\partial \dot{q}_j} = 0$. Also $\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = 0$ and $\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij} = 1$ if $i = j$.

Therefore for above equation,

$$\begin{aligned} \text{R.H.S.} &= \frac{d}{dt} \left[\frac{\partial F}{\partial q_j} \right] - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_j} \right) \dot{q}_j \quad \text{taking } i = j \\ &= \frac{\partial}{\partial q_j} \left(\frac{dF}{dt} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_j} \right) \dot{q}_j \\ &= \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_j} \right) \dot{q}_j \\ &= \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_j} \right) \dot{q}_j + \frac{\partial F}{\partial q_j} \frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial t} \right) - \frac{\partial}{\partial q_j} \left(\frac{\partial F}{\partial q_j} \right) \dot{q}_j \\ &= 0 \end{aligned}$$

But under Lagrangian Gauge transformation, the canonical momenta and the Jacobi's integral change.

Let

$$L' = L + \frac{dF}{dt}$$

then momenta

$$\begin{aligned} p'_j &= \frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j} \left(\frac{\partial F}{\partial t} + \dot{q}_i \frac{\partial F}{\partial q_i} \right) \\ &= p_j + \frac{\partial F}{\partial q_j} \end{aligned}$$

and Jacobi's integral

$$\begin{aligned} J' &= \dot{q}_j \frac{\partial L'}{\partial \dot{q}_j} - L' = J + \dot{q}_j \frac{\partial F}{\partial q_j} - \frac{dF}{dt} \\ &= J - \frac{\partial F}{\partial t} \end{aligned}$$

2.14. INVARIANCE OF LAGRANGE'S EQUATIONS UNDER GENERALISED COORDINATE TRANSFORMATIONS (POINT TRANSFORMATIONS)

Let q_j and Q_j denote the coordinates of the two sets (called old and new set) and corresponding Lagrangian as

$$L = L(q_j, \dot{q}_j, t) \quad \text{and} \quad L' = L'(Q_i, \dot{Q}_i, t)$$

then equations of transformation between them can be written

$$q_j = q_j(Q_1 \dots Q_n, t)$$

or inside out equations as

$$Q_i = Q_i(q_1 \dots q_n, t)$$

From above equations, we have

$$\dot{q}_j = \frac{\partial q_j}{\partial t} + \frac{\partial q_j}{\partial Q_i} \dot{Q}_i$$

which gives

$$\frac{\partial \dot{q}_j}{\partial \dot{Q}_i} = \frac{\partial q_j}{\partial Q_i}$$

Further

$$\frac{d}{dt} \left(\frac{\partial q_j}{\partial Q_i} \right) = \frac{\partial \dot{q}_j}{\partial \dot{Q}_i}$$

and

$$0 = \frac{\partial t}{\partial \dot{Q}_i} = \frac{\partial t}{\partial \dot{Q}_i} = \frac{\partial q_j}{\partial \dot{Q}_i}$$

Such transformations are called point transformations by definition. Now to show invariance, we start with Lagrange's equation in new coordinates

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{Q}_i} \right) - \frac{\partial L'}{\partial Q_i} &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \dot{Q}_i} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{Q}_i} + \frac{\partial L}{\partial t} \frac{\partial t}{\partial \dot{Q}_i} \right) \\ &\quad - \left(\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial Q_i} + \frac{\partial L}{\partial t} \frac{\partial t}{\partial Q_i} \right) \\ &= \frac{d}{dt} \left(0 + \frac{\partial L}{\partial \dot{q}_j} \frac{\partial q_j}{\partial \dot{Q}_i} + 0 \right) - \left[\frac{\partial L}{\partial q_j} \frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \left(\frac{\partial q_j}{\partial Q_i} \right) + 0 \right] \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) \frac{\partial q_j}{\partial \dot{Q}_i} + \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \left(\frac{\partial q_j}{\partial \dot{Q}_i} \right) - \frac{\partial L}{\partial q_j} \left(\frac{\partial q_j}{\partial Q_i} \right) - \frac{\partial L}{\partial \dot{q}_j} \frac{d}{dt} \left(\frac{\partial q_j}{\partial Q_i} \right) \\ &= \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] \frac{\partial q_j}{\partial \dot{Q}_i} \end{aligned}$$

Thus if the most general Euler-Lagrange's equation of motion with cartesian components of non-potential forces as F'_k are valid in terms of the old set of coordinates, that is, if

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = F'_k \frac{\partial x_k}{\partial q_j} \quad \text{for every } q_j$$

then

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{Q}_i} \right) - \frac{\partial L'}{\partial Q_i} = F'_k \frac{\partial x_k}{\partial Q_i}$$

is also valid for every Q_i coordinate. Therefore, Lagrange's equation of motion, in their most general form, remain invariant in form under the most general (that is, time dependent) generalised coordinate transformation.

2.15. CONCEPT OF SYMMETRY : HOMOGENEITY AND ISOTROPY :

If any system or any function representing a property of the system does not change under some operation carried on the system, the system is said to possess a *symmetry with respect to that operation*. For example, when a cylinder is rotated about its axis by any angle then its apparent shape does not change. The cylinder is said to have rotational symmetry about its axis. In the present context, we shall often refer to a closed system. A closed system, by definition is the one that is not acted upon by any field of force, whose source is external to the system. A closed system is described as any other system by its Lagrangian. In particular, we require that the Lagrangian of a closed system should be invariant under the operations of translation and rotation in space and that it should not depend explicitly on time, i.e. $(\partial L / \partial t) = 0$. Now we discuss the invariance of physical properties of space and time under certain kinds of symmetry operations.

(✓) *Homogeneity of time and conservation of energy :*

If for any arbitrary displacement of the origin of time, the physical properties of any closed system remain unaffected, we say that time is homogeneous.

As mentioned earlier, for closed system $(\partial L / \partial t) = 0$ and there is no external force. Therefore both kinetic and potential energies are constant. That is

$$J = E = T + V = \text{Constant.}$$

Hence, due to homogeneity of time, total energy is conserved for any closed system.

✓(2) Homogeneity of space and conservation of linear momentum :

If for any arbitrary displacement of the origin of any reference frame the physical properties of all closed systems remain unaffected, we say that space is homogeneous.

The Lagrangian of a closed system should not change due to any arbitrarily small uniform translation of all particles.

As for closed system external force is zero, we can refer the case to art 2.11. As generalised force $Q_j = 0$, we can take that arbitrary small uniform translation of all particles, in effect, amounts to the translational coordinate being a cyclic one. Consequently, the discussion given there is equally valid here. The inference is that the total linear momentum of any closed system is conserved due to the homogeneity of space.

(3) Isotropy of space and conservation of angular momentum :

If for any arbitrary rotation about the origin of any reference frame the physical properties of any closed system remain unaffected, we say that space is isotropic.

The Lagrangian for a closed system should not change due to any arbitrary small rotation of reference frame about some arbitrary direction. For a closed system external torque is zero and therefore we refer to art 2.11, fig 2.14 where a small rotation, dq_j is affected about the axis of rotation along which the unit vector is $\mathbf{n}$. It has been shown there that

$$\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \mathbf{r}_i}{\partial \theta} = \mathbf{n} \times \mathbf{r}_i$$

and we come to conclusion that if rotation coordinate is cyclic, angular momentum is conserved. ✓

Coordinate symmetry operations, which result in the existence of cyclic coordinates, leave the description of the motion of the system invariant. In other words, Lagrangian of a closed system must be invariant under the translations and rotations in the space. This is so because these are the symmetry operations for the Lagrangian. Each of these symmetries corresponds to a conservation law, giving rise to a conserved quantity. It turns out that every symmetry in the Lagrangian corresponds to a conservation law. If the system is spherically symmetric, for example, we can say without further ado that all components of angular momentum are conserved.

2.16. INVARIANCE OF LAGRANGE'S EQUATIONS UNDER GALILEAN TRANSFORMATION :

Consider two inertial frames S and S' . S' is moving with constant velocity $\mathbf{v}_0$ with respect to frame S . Obviously, Newton's law $\mathbf{F} = m\mathbf{a}$ will be valid in both frames of reference. The Galilean transformations are :

for displacement

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t$$

for velocity

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$$

If L is Lagrangian in S then

$$L = \frac{1}{2} m v^2 - V$$

When we write Lagrangian L' in S' then only $\mathbf{v}$ will change to $\mathbf{v}'$ but not the potential energy V because it is a function of interparticle distance i.e. $(\mathbf{r}_2 - \mathbf{r}_1) = (\mathbf{r}'_2 - \mathbf{r}'_1)$ at all instants. Therefore

$$\begin{aligned} L' &= \frac{1}{2} m v'^2 - V \\ &= \frac{1}{2} (\mathbf{v} - \mathbf{v}_0)^2 - V \\ &= \frac{1}{2} m v^2 - V - m \mathbf{v} \cdot \mathbf{v}_0 + \frac{1}{2} m v_0^2 \end{aligned}$$

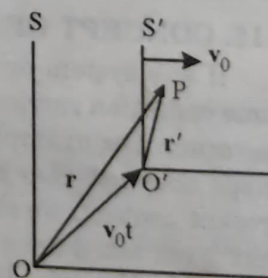


Fig. 2.15.

$$\begin{aligned}
 &= L + \frac{d}{dt} \left(\frac{1}{2} m \mathbf{v}_0^2 - m \mathbf{v}_0 \cdot \mathbf{r} \right) \\
 &= L + \frac{dF}{dt}(\mathbf{r}, t)
 \end{aligned}$$

Now refer to art 2.13, $F(\mathbf{r}, t)$ is recognised as gauge function for Lagrangian and we have shown there that Lagrange's equations retain their form in L or $\left(L + \frac{dF}{dt} \right)$. This proves that Lagrange's equations preserve their form in S' also.

2.17. SUPERIORITY OF LAGRANGIAN APPROACH OVER NEWTONIAN APPROACH

Classical mechanics concerns itself with the key problem of determining the motion of a system of N -particles acted on by arbitrary but known forces to arbitrary but known constraints and possessing known initial positions and velocities in the primary inertial system. Such a problem can be phrased equally well in both ways i.e., equations of Newton and equations of Lagrange, but the latter way has the following advantages:

- (a) Lagrange's equations of motion are invariant in form with respect to co-ordinate transformation.
- (b) They put emphasis on energy and work rather than on force.
- (c) They are expressed in generalised co-ordinates which do not have to refer directly to the position or displacements of actual particles.
- (d) As stated in point (a) Lagrange's equations of motion neither change their form under point transformation nor do they change if we add to the Lagrangian a total time derivative of any arbitrary function $F(q, t)$. The latter is called the Lagrangian gauge-symmetry of Euler-Lagrange's equations of motion.
- (e) It turns out that every symmetry (with respect to arbitrary translation or rotation or passage of time) in the Lagrangian corresponds to a conservation law. Thus symmetries in Lagrangian have very important consequences.

In problems in which the particle motions are not readily detectable, Lagrangian approach is of great value on account of involving the concept of work and energy.

WORKED EXAMPLES

1. Find the equation of motion of one dimensional harmonic oscillator using Hamilton's principle.

The Lagrangian for one dimensional harmonic oscillator is (cf. art. 2.9).

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2.$$

Then on applying Hamilton's principle, we get

$$\delta \int \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 \right) dt = 0$$

Let

$$f(x, \dot{x}, t) = \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 \right),$$

then $\int f(x, \dot{x}, t) dt$ should be extremum for which f must satisfy the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) - \frac{\partial f}{\partial x} = 0.$$

Further,

$$\frac{\partial f}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial f}{\partial x} = -kx,$$

so that the Euler-Lagrange equation takes the form

$$m\ddot{x} + kx = 0.$$

and is equation of motion for one dimensional harmonic oscillator.

2. A particle of mass m falls a given distance z_0 in time $t_0 = \left(\frac{2z_0}{g}\right)^{1/2}$ and the distance travelled in time t is given by $z = at + bt^2$, where constants a and b are such that the time t_0 is always the same.

Show that the integral $\int_0^{t_0} L dt$ is extremum for real values of the coefficients only when $a = 0$ and $b = g/2$.

If the distance travelled in time t is z , the kinetic energy of particle will be $\frac{1}{2}m\dot{z}^2$ and the potential energy will be $-mgz$. Therefore, Lagrangian will be

$$L = T - V = \frac{1}{2}m\dot{z}^2 + mgz.$$

Then on applying Hamilton's principle, we get

$$\delta \int \left(\frac{1}{2}m\dot{z}^2 + mgz \right) dt = 0,$$

or path $\int f(z, \dot{z}, t) dt$ should be extremum, where $f(z, \dot{z}, t)$ is equal to $(\frac{1}{2}m\dot{z}^2 + mgz)$. For the path to be extremum, this function $f(z, \dot{z}, t)$ must satisfy the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{z}} \right) - \frac{\partial f}{\partial z} = 0. \quad \dots (1)$$

Further, from the expression for $f(z, \dot{z}, t)$ we find

$$\frac{\partial f}{\partial \dot{z}} = m\dot{z} \quad \text{and} \quad \frac{\partial f}{\partial z} = mg,$$

so that eq. (1) takes the form

$$m\ddot{z} - mg = 0$$

But

$$z = at + bt^2,$$

so that $\ddot{z} = 2b$ which when substituted in equation (2) gives

$$b = \frac{g}{2}. \quad \dots (3)$$

Also when $t = t_0$, $z = z_0$, so that

$$z_0 = at_0 + bt_0^2.$$

On putting $z_0 = \frac{gt_0^2}{2}$ and $b = \frac{g}{2}$, we get

$$\frac{gt_0^2}{2} = at_0 + \frac{gt_0^2}{2},$$

which gives

$$a = 0.$$

Therefore the path will be extremum or the integral $\int_0^{t_0} L dt$ is extremum when $a = 0$ and

$$b = \frac{g}{2}.$$

3. A particle of mass M moves on a plane in the field of force given by

$$\mathbf{F} = -\hat{\mathbf{r}} kr \cos \theta,$$

where k is constant and $\hat{\mathbf{r}}$ is the radial unit vector.

(a) Will the angular momentum of the particle about the origin be conserved? Justify your statement.

(b) Obtain the differential equation of the orbit of the particle.

Suppose (r, θ) are the polar co-ordinates of the particle of mass M in the given plane. The kinetic energy of the particle will be

$$T = \frac{1}{2} M(\dot{r}^2 + r^2 \dot{\theta}^2). \quad \dots (1)$$

(a) To show it, let us write the Lagrange's equation of motion in terms of θ , i.e.,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} = Q_{\theta}. \quad \dots (2)$$

From equation (1)

$$\left. \begin{aligned} \frac{\partial T}{\partial \dot{\theta}} &= Mr^2 \dot{\theta}, \\ \frac{\partial T}{\partial \theta} &= 0 \end{aligned} \right\}$$

and since the force is only radial, $Q_{\theta} = 0$. Equation (2), therefore, becomes

$$\frac{d}{dt} (Mr^2 \dot{\theta}) = 0.$$

or

$$Mr^2 \dot{\theta} = \text{a constant.}$$

The left hand side of above equation is the angular momentum of the particle about the origin and is, therefore, conserved.

(b) Differential equation of the orbit is a relation in co-ordinates r and θ . We write the Lagrange's equation in term of generalised co-ordinate r , i.e.,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}} \right) - \frac{\partial T}{\partial r} = Q_r. \quad \dots (3)$$

From eq. (1),

$$\left. \begin{aligned} \frac{\partial T}{\partial \dot{r}} &= M\dot{r} \\ \frac{\partial T}{\partial r} &= Mr\dot{\theta}^2 \end{aligned} \right\}$$

Also $Q_r = F_r = -kr \cos \theta$. Therefore equation (3) becomes

$$\frac{d}{dt} (M\dot{r}) - Mr\dot{\theta}^2 = -kr \cos \theta$$

or

$$M\ddot{r} - Mr\dot{\theta}^2 + kr \cos \theta = 0. \quad \dots (4)$$

Eq. (4) is the desired differential equation of the orbit of the particle.

4. A bead slides on a wire in the shape of a cycloid described by equations

$$x = a(\theta - \sin \theta)$$

$$y = a(1 + \cos \theta)$$

where $0 \leq \theta \leq 2\pi$.

Find (a) the Lagrangian function, and (b) the equation of motion. Neglect friction between the bead and the wire.

(a) Let $P(x, y)$ be the position of the bead of the mass m at any instant t . The path of bead is a cycloid represented by equations

$$\left. \begin{aligned} x &= a(\theta - \sin \theta), \\ y &= a(1 + \cos \theta), \end{aligned} \right\}$$

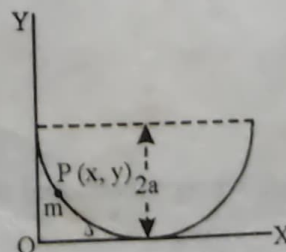


Fig. 2.16.

The kinetic energy of the bead will be

$$\begin{aligned} T &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2} m (a^2 (\dot{\theta} - \cos \theta \dot{\theta})^2 + a^2 (-\sin \theta \dot{\theta})^2) \\ &= ma^2 \dot{\theta}^2 (1 - \cos \theta). \end{aligned}$$

The potential energy of the bead, taking x -axis as the reference level will be

$$V = mgy = mga (1 + \cos \theta).$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= ma^2 \dot{\theta}^2 (1 - \cos \theta) - mga (1 + \cos \theta). \end{aligned} \quad \dots (1)$$

(b) The Lagrange's equation of motion is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0. \quad \dots (2)$$

From eq. (1) we find

$$\frac{\partial L}{\partial \dot{\theta}} = 2ma^2 \dot{\theta} (1 - \cos \theta)$$

and

$$\frac{\partial L}{\partial \theta} = ma^2 \dot{\theta}^2 \sin \theta + mga \sin \theta$$

which when substituted in eq. (2) gives

$$\frac{d}{dt} [2ma^2 \dot{\theta} (1 - \cos \theta)] - [ma^2 \dot{\theta}^2 \sin \theta + mga \sin \theta] = 0$$

$$\text{or } 2ma^2 \ddot{\theta} (1 - \cos \theta) + 2ma^2 \dot{\theta} \sin \theta \dot{\theta} - ma^2 \dot{\theta}^2 \sin \theta - mga \sin \theta = 0$$

$$\text{or } \ddot{\theta} (1 - \cos \theta) + \frac{1}{2} \sin \theta \dot{\theta}^2 - \frac{g}{2a} \sin \theta = 0. \quad \dots (3)$$

Equation (3) gives the required equation of motion.

5. A particle of mass m is projected with initial velocity u at an angle α with the horizontal. Use Lagrange's equations to describe the motion of the projectile. The resistance of the air may be neglected.

Let $P(x, y)$ be the position of the particle at any instant t in XY -plane with x -axis as horizontal. The kinetic energy of the particle will be

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

and the potential energy, taking x -axis as the reference level, will be

$$V = mgy$$

The Lagrangian is then

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

giving

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x}, & \frac{\partial L}{\partial \dot{y}} &= m\dot{y} \\ \frac{\partial L}{\partial x} &= 0, & \frac{\partial L}{\partial y} &= -mg \end{aligned}$$

which when substituted in Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0,$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0,$$

yield

$$\left. \begin{aligned} \dot{x}' &= 0 \\ \dot{y}' + g &= 0 \end{aligned} \right\}$$

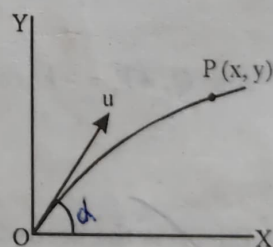


Fig. 2.17.

Integrating the above equations twice, we get

$$x = At + B, \quad \dots (1)$$

$$y = -\frac{1}{2}gt^2 + Ct + D, \quad \dots (2)$$

where A, B, C and D are constants of integration which can be evaluated by applying the following boundary conditions :

At $t = 0$,

$$\left. \begin{aligned} x &= 0 \\ \dot{x} &= u \cos \alpha \end{aligned} \right\} \text{for equation (1),}$$

and

$$\left. \begin{aligned} y &= 0 \\ \dot{y} &= u \sin \alpha \end{aligned} \right\} \text{for equation (2).}$$

On applying these to their respective equations, we get

$$x = (u \cos \alpha)t, \quad \dots (3)$$

$$y = (u \sin \alpha)t - \frac{1}{2}gt^2, \quad \dots (4)$$

Eliminating t from eq. (4) with help of eq. (3), we get

$$y = x \tan \alpha - \frac{1}{2}g \frac{x^2}{u^2 \cos^2 \alpha}$$

which is the desired equation of the path followed by the projectile and is a parabola.

6. A particle is constrained to move two dimensionally on a smooth inclined plane making an angle ($\alpha < 90^\circ$) to the horizontal. Assume that the line of intersection of the inclined plane with the horizontal plane is the x -axis and a line on the inclined plane drawn perpendicular to the x -axis is the y -axis.

- Obtain an expression for Lagrangian of the particle in terms of the co-ordinates (x, y) .
- Solve the Lagrange's equations of motion and show that the trajectory of the particle on the inclined plane will be a parabola, if the particle was initially at $x = 0, y = 1$ with an initial velocity v_0 along the line $x = y$.

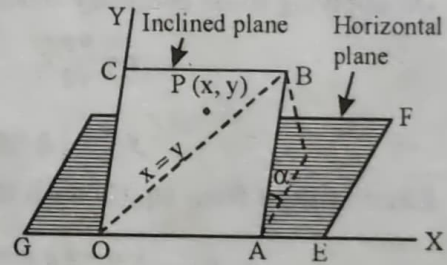


Fig. 2.18.

In fig. (2.18) $OABC$ is the plane inclined at angle α to the horizontal plane GEF . The line of intersection of the two planes is OA and has been taken as the x -axis. y -axis is drawn on the inclined plane and is perpendicular to x -axis.

Let at any instant t , $P(x, y)$ be the position of the particle of mass m on the inclined plane. Then kinetic energy of the particle will be

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

and the potential energy

$$\begin{aligned} V &= mg \times \text{height of the particle } P \text{ from the horizontal plane} \\ &= mgy \sin \alpha. \end{aligned}$$

(a) Then Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \sin \alpha,$$

giving

$$\left. \begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\dot{x}, & \frac{\partial L}{\partial \dot{y}} &= m\dot{y}; \\ \frac{\partial L}{\partial x} &= 0, & \frac{\partial L}{\partial y} &= -mg \sin \alpha. \end{aligned} \right\} \quad \dots (1)$$

(b) The Lagrange's equations of motion for such a system will be

$$\left. \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} &= 0, \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} &= 0. \end{aligned} \right\}$$